Free affine actions on the plane with discrete orbits are properly discontinuous

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Abstract

Let $\Gamma \leqslant \mathrm{Aff}(\mathbb{R}^2)$ be a group of affine transformations of the plane acting *freely* and with *discrete orbits* (in the subspace topology). We give an elementary, self-contained proof that Γ then acts *properly discontinuously*, so that the orbit space $\mathbb{R}^2/\Gamma$ is a (Hausdorff, complete flat affine) surface. In general a free action with discrete orbits need not be proper. Whether, in every dimension, a free affine action with discrete orbits must be proper is Charlap's Open Problem 1.1, for which only an unpublished proof-sketch of Kuiper for $n = 2$ had been recorded; the analogous implication in full topological generality is Kapovich's Question 14, which remains open. The classification of the resulting groups into Kuiper's three models is classical; the contribution here is a complete, elementary proof of the passage from the weak hypothesis to proper discontinuity in dimension two. The only external inputs are the existence of a recurrent point for a homeomorphism of a compact metric space and Weyl's equidistribution theorem.

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1 Introduction

Let $\mathrm{Aff}(\mathbb{R}^n) = \mathrm{GL}_n(\mathbb{R}) \ltimes \mathbb{R}^n$ be the group of (invertible) affine transformations of $\mathbb{R}^n$; an element is a pair $\gamma = (L_\gamma, v_\gamma)$ with $L_\gamma \in \mathrm{GL}_n(\mathbb{R})$, $v_\gamma \in \mathbb{R}^n$, acting by

$$\gamma \cdot x \;=\; L_\gamma x + v_\gamma, \qquad x \in \mathbb{R}^n.$$

A continuous action of a group G on a topological space X is said to have *discrete orbits* if every orbit Gx is a discrete subset of X in the subspace topology (equivalently, every orbit point is isolated in its orbit). This is the weakest link in a chain of strengthenings,

$$\text{properly discontinuous} \implies \text{wandering} \implies \text{orbits closed and discrete} \implies \text{orbits discrete},$$

in which no implication reverses in general; the failures relevant here are Kapovich's Example 12 below (wandering $\not\Rightarrow$ properly discontinuous) and the last arrow, where an orbit can be discrete as a subspace yet fail to be closed - the very gap this note must close. (Proper discontinuity is meant in the sense of Definition 2; an action is *wandering*, in the sense of Kapovich [5], if every point has a neighbourhood U with $\{g : gU \cap U \neq \varnothing\}$ finite, and such an action automatically has all orbits closed and discrete [5, Lemma 13].) Already wandering does not imply proper discontinuity: Kapovich [5, Example 12] exhibits a free wandering action that is not proper, namely $\mathbb{Z}$ acting on the punctured plane $\mathbb{R}^2 \setminus \{0\}$ by $(x, y) \mapsto (2x,\, y/2)$, whose orbits are closed and discrete but whose quotient is not Hausdorff. The hypothesis of the present note is weaker still than wandering: we do not assume orbits closed, so a priori an orbit could be discrete *as a subspace* while accumulating onto a *different* orbit.

For affine actions, the question whether a free action with discrete orbits on $\mathbb{R}^n$ must be properly discontinuous is, in the theory of Bieberbach groups and flat manifolds, Charlap's *Open Problem 1.1* [3, p. 4]:[1]

> *If $G \leqslant \mathrm{Aff}(\mathbb{R}^n)$ acts freely with discrete orbits on $\mathbb{R}^n$, then the orbit space $\mathbb{R}^n/G$ is an n-manifold.*

The substance, in every dimension, is to upgrade "discrete orbits" to "properly discontinuous"; for a free properly discontinuous action the quotient is automatically a manifold [4]. Without the affineness assumption, the analogous implication - that a free continuous action on a manifold whose quotient is again a manifold must be proper - is Kapovich's Question 14 [5, Question 14], open in full topological generality. Kapovich's punctured-plane example shows the upgrade can fail on a general surface. We prove that on the *full* affine plane it does hold, giving an elementary, self-contained proof of the case $n = 2$ of Charlap's Open Problem 1.1 - for which [3] records only an unpublished proof-sketch of Kuiper.

Theorem 1. *Let $\Gamma \leqslant \mathrm{Aff}(\mathbb{R}^2)$ act freely on $\mathbb{R}^2$ and suppose that every orbit Γp ($p \in \mathbb{R}^2$) is discrete in the subspace topology. Then Γ acts properly discontinuously on $\mathbb{R}^2$. Consequently the quotient map $\mathbb{R}^2 \to \mathbb{R}^2/\Gamma$ is a covering map and $\mathbb{R}^2/\Gamma$ is a topological surface; it carries a complete flat affine structure.*

The structure obtained along the way reproduces, up to finite index and an affine change of coordinates, Kuiper's three plane models [6, 1]

$$T = \{(x, y) \mapsto (x+t,\ y+s)\}, \quad P = \{(x, y) \mapsto (x+sy+t,\ y+s)\}, \quad H = \{(x, y) \mapsto (e^t x,\ y+t)\},$$

which act properly on $\mathbb{R}^2$. Thus the present note does not establish a new classification - that is classical [6, 1] - but rather gives a self-contained, elementary proof that in dimension two the weak hypothesis "free with discrete orbits" already forces proper discontinuity, settling the case $n = 2$ of Charlap's Open Problem 1.1 [3].

Strategy. The proof has two halves. The crucial first step (Proposition 4) shows that freeness already turns the weak orbit hypothesis into a strong statement *in the affine group*: Γ is discrete and closed in $\mathrm{Aff}(\mathbb{R}^2)$. The second half (Sections 3–5) is a classification: using that every nonidentity element of Γ is fixed-point free, we triangularise the linear parts and then pin down the possible affine groups, showing each lands inside a closed subgroup of $\mathrm{Aff}(\mathbb{R}^2)$ that acts *properly*. Combined with discreteness in $\mathrm{Aff}(\mathbb{R}^2)$, this yields proper discontinuity (Lemma 7).

Conventions and provenance. The elementary fixed-point lemma (Lemma 8) is Abels' [1, Lemma 6.1]; the three plane models and their classification under proper discontinuity are due to Kuiper [6] and recorded in [1, §6]. The present argument is self-contained: beyond Lemma 8 (which we reprove), the only external inputs are Weyl's equidistribution theorem [7] and the

[1] Charlap calls a subgroup $\pi \leqslant \mathrm{A}_n$ *discontinuous* if all its orbits are discrete ([3, Definition 1.4]). His Exercise 1.6 asks the reader to prove the manifold conclusion for subgroups of the isometry group M_n (where the linear parts lie in the compact group O_n), and Open Problem 1.1 asks whether the same holds "if π is merely a subgroup of A_n". He identifies the crux ([3, Remark ii]) as showing that "a subgroup of A_n that acts discontinuously (discrete orbits) acts properly discontinuously"; he notes ([3, Remark i]) that the non-compactness of GL_n makes this delicate, records "an unpublished sketch of a proof for dimension 2 due to N. Kuiper", and adds that the result "seems to have been tacitly assumed in a number of papers on locally affine manifolds". This unpublished dimension-2 sketch is distinct from Kuiper's *published* classification of locally affine surfaces [6] (the proper-discontinuity classification recorded in [1, Proposition 6.5]), which presupposes proper discontinuity.

existence of recurrent points for homeomorphisms of compact metric spaces [2]. Throughout, "properly discontinuous" means:

Definition 2. An action of a group Γ on a locally compact Hausdorff space X by homeomorphisms is *properly discontinuous* if for every compact $K \subseteq X$ the set $\{\gamma \in \Gamma : \gamma K \cap K \neq \varnothing\}$ is finite.

2 Discreteness in the affine group

We topologise $\mathrm{Aff}(\mathbb{R}^2)$ as the open subset $\mathrm{GL}_2(\mathbb{R}) \times \mathbb{R}^2$ of $M_2(\mathbb{R}) \times \mathbb{R}^2 \cong \mathbb{R}^6$; multiplication and inversion are continuous, and the evaluation map $\mathrm{Aff}(\mathbb{R}^2) \times \mathbb{R}^2 \to \mathbb{R}^2$, $(\gamma, p) \mapsto \gamma p$, is continuous.

Lemma 3 (Accumulation lemma). *Let Γ act freely on $\mathbb{R}^2$ with all orbits discrete, and fix $p \in \mathbb{R}^2$. Then there is no sequence $(g_j)_{j \geq 1}$ in $\Gamma \setminus \{\mathrm{id}\}$ with $g_j p \to p$.*

Proof. If $g_j \neq \mathrm{id}$ then g_j is fixed-point free (freeness), so $g_j p \neq p$. Were $g_j p \to p$, then every neighbourhood of p would contain a point $g_j p \in \Gamma p$ different from p, so p would not be isolated in its orbit Γp - contradicting discreteness. □

Remark. This is the only point where the two hypotheses are coupled: freeness converts the bare assumption "orbit points are isolated" into the dynamical statement that no sequence of nonidentity elements can move a point back to itself. Only isolatedness is used; no closedness of orbits in $\mathbb{R}^2$ is assumed. Proposition 4 below promotes this to discreteness of Γ *in* $\mathrm{Aff}(\mathbb{R}^2)$.

Proposition 4. *Let $\Gamma \leqslant \mathrm{Aff}(\mathbb{R}^2)$ act freely with all orbits discrete. Then*

$$\Gamma \cap C \text{ is finite for every compact } C \subseteq \mathrm{Aff}(\mathbb{R}^2). \tag{1}$$

In particular Γ is a discrete and closed subgroup of $\mathrm{Aff}(\mathbb{R}^2)$.

Proof. Suppose not: some compact C contains infinitely many distinct elements $\gamma_1, \gamma_2, \dots \in \Gamma$. By compactness, after passing to a subsequence, $\gamma_i \to A$ for some $A \in C \subseteq \mathrm{Aff}(\mathbb{R}^2)$; in particular the limit A is again affine. Since inversion is continuous on $\mathrm{Aff}(\mathbb{R}^2)$, $\gamma_i^{-1} \to A^{-1}$, hence

$$\delta_i \;:=\; \gamma_i^{-1}\gamma_{i+1} \;\longrightarrow\; A^{-1}A = \mathrm{id}.$$

The γ_i are pairwise distinct, so $\delta_i \neq \mathrm{id}$; and for any fixed $p \in \mathbb{R}^2$, continuity of the evaluation map gives $\delta_i p \to p$. Thus (δ_i) is a sequence in $\Gamma \setminus \{\mathrm{id}\}$ with $\delta_i p \to p$, contradicting Lemma 3.

For the final assertion: if $\gamma_i \to A$ in $\mathrm{Aff}(\mathbb{R}^2)$ with $\gamma_i \in \Gamma$, then $\{A\} \cup \{\gamma_i\}$ is compact, so by (1) only finitely many γ_i are distinct; a convergent sequence taking finitely many values is eventually constant, so $A = \gamma_i \in \Gamma$ for large i. Hence Γ is closed, and has no accumulation points, i.e. it is discrete. □

The following consequence of (1) is used repeatedly: it is the precise tool that converts "a sequence of group elements converges" into "it is eventually constant", and it is what guards against a convergent sequence of *distinct* elements escaping to a limit outside Γ.

Corollary 5. *If a sequence $(g_n)_{n \geq 1}$ in Γ converges in $\mathrm{Aff}(\mathbb{R}^2)$, then it is eventually constant.*

Proof. The set $\{g_n : n \geq 1\} \cup \{\lim_n g_n\}$ is compact, so by (1) it contains only finitely many elements of Γ; hence (g_n) takes finitely many values. A convergent sequence taking finitely many values is eventually constant. □

We isolate two further soft consequences of Definition 2 that will be used to close each branch of the classification.

Lemma 6. *Let $\Gamma' \leqslant \Gamma$ be a subgroup of finite index. If Γ' acts properly discontinuously on X, then so does Γ.*

Proof. Write $\Gamma = \bigsqcup_{i=1}^{k} \gamma_i \Gamma'$. Let $K \subseteq X$ be compact and set $K' = K \cup \bigcup_{i=1}^{k} \gamma_i^{-1} K$, which is compact. If $\gamma = \gamma_i \gamma' \in \Gamma$ (with $\gamma' \in \Gamma'$) satisfies $\gamma K \cap K \neq \varnothing$, then $\gamma' K \cap \gamma_i^{-1} K \neq \varnothing$; since $K \subseteq K'$ and $\gamma_i^{-1} K \subseteq K'$, this gives $\gamma' K' \cap K' \neq \varnothing$. The set of such γ' is finite by hypothesis, so $\{\gamma : \gamma K \cap K \neq \varnothing\} \subseteq \bigcup_i \gamma_i \{\gamma' : \gamma' K' \cap K' \neq \varnothing\}$ is finite. $\square$

Lemma 7. *Let $G_0 \leqslant \mathrm{Aff}(\mathbb{R}^2)$ be a closed subgroup that acts* properly *on $\mathbb{R}^2$, in the sense that for every compact $K \subseteq \mathbb{R}^2$ the set $G_0(K) := \{g \in G_0 : gK \cap K \neq \varnothing\}$ is compact. If $\Gamma \leqslant G_0$ satisfies (1), then Γ acts properly discontinuously on $\mathbb{R}^2$.*

Proof. For compact K, $\{\gamma \in \Gamma : \gamma K \cap K \neq \varnothing\} = \Gamma \cap G_0(K)$. Now $G_0(K)$ is compact in G_0, hence compact in $\mathrm{Aff}(\mathbb{R}^2)$, so by (1) its intersection with Γ is finite. $\square$

3 The linear normal form

Lemma 8 (Fixed-point lemma; Abels [1, Lemma 6.1])**.** *If $\gamma \in \mathrm{Aff}(\mathbb{R}^2)$ has no fixed point, then 1 is an eigenvalue of L_γ.*

Proof. If $1 \notin \mathrm{spec}(L_\gamma)$ then $I - L_\gamma$ is invertible and $x_0 = (I - L_\gamma)^{-1} v_\gamma$ solves $L_\gamma x_0 + v_\gamma = x_0$, a fixed point. $\square$

Since Γ acts freely, *every* nonidentity $\gamma \in \Gamma$ is fixed-point free, so $1 \in \mathrm{spec}(L_\gamma)$. Consequently the linear part $\bar{\Gamma} := \{L_\gamma : \gamma \in \Gamma\}$, which is a subgroup of $\mathrm{GL}_2(\mathbb{R})$ (the linear part is a homomorphism), has the property that *every* element has 1 as an eigenvalue (for $L_\gamma = I$ this is trivial).

Lemma 9. *Let $\bar{\Gamma} \leqslant \mathrm{GL}_2(\mathbb{R})$ be a subgroup such that every element of $\bar{\Gamma}$ has 1 as an eigenvalue. Then, after a linear change of coordinates, $\bar{\Gamma}$ is contained in one of the two groups*

$$\mathcal{U}_1 = \left\{ \begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix} : b \neq 0 \right\}, \qquad \mathcal{U}_2 = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a \neq 0 \right\}.$$

Proof. Every $M \in \bar{\Gamma}$ has 1 as an eigenvalue, and since its characteristic polynomial is real of degree 2, the other eigenvalue is real and equals $\mathrm{tr}(M) - 1$. Hence each M is either diagonalisable with eigenvalues $1 \neq \lambda$, or unipotent (both eigenvalues 1). The two cases below - whether or not some element is of the first kind - are therefore exhaustive. In each case we show $\bar{\Gamma}$ is simultaneously triangularisable; a final union-of-subgroups argument then separates $\mathcal{U}_1$ from $\mathcal{U}_2$.

Case 1: some $A \in \bar{\Gamma}$ has a second eigenvalue $\lambda \neq 1$. Choose a basis of eigenvectors so that $A = \mathrm{diag}(1, \lambda)$; testing each element against both itself and its product with this diagonal probe A turns the eigenvalue hypothesis into a pair of linear constraints. Let $B = \left(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right) \in \bar{\Gamma}$. Here $B - I = \left(\begin{smallmatrix} p-1 & q \\ r & s-1 \end{smallmatrix}\right)$ and, since $AB = \left(\begin{smallmatrix} p & q \\ \lambda r & \lambda s \end{smallmatrix}\right)$, also $AB - I = \left(\begin{smallmatrix} p-1 & q \\ \lambda r & \lambda s-1 \end{smallmatrix}\right)$. Both B and AB have 1 as an eigenvalue, i.e. $\det(B - I) = 0$ and $\det(AB - I) = 0$:

$$(p-1)(s-1) - qr = 0, \qquad (p-1)(\lambda s - 1) - \lambda qr = 0.$$

Subtracting λ times the first equation from the second gives $(p-1)\big[(\lambda s - 1) - \lambda(s-1)\big] = (p-1)(\lambda - 1) = 0$, so $p = 1$ (as $\lambda \neq 1$); the first equation then forces $qr = 0$. Thus every $B \in \bar{\Gamma}$ has the form $\left(\begin{smallmatrix} 1 & q \\ r & s \end{smallmatrix}\right)$ with $qr = 0$, i.e. is upper or lower triangular. We cannot have both types nontrivially: if $B = \left(\begin{smallmatrix} 1 & q \\ 0 & s \end{smallmatrix}\right)$ with $q \neq 0$ and $C = \left(\begin{smallmatrix} 1 & 0 \\ r & t \end{smallmatrix}\right)$ with $r \neq 0$, then $(BC)_{11} = 1 + qr \neq 1$, contradicting that every element of $\bar{\Gamma}$ has $(1,1)$-entry 1. Hence all elements of $\bar{\Gamma}$ are simultaneously upper triangular, or all lower triangular; in the latter case conjugating by the coordinate swap $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ makes them all upper triangular.

Case 2: every nonidentity element of $\bar{\Gamma}$ is unipotent (both eigenvalues 1). A nonzero 2×2 matrix N with $N^2 = 0$ has rank 1, so $N = vw^\top$ for a nonzero column v and row $w^\top$; then $N^2 = (w^\top v)\, vw^\top$, which vanishes iff $w^\top v = 0$. The eigenline of $U = I + N$ (the kernel of N) is $\mathbb{R}v$. For two such $U_i = I + N_i$ $(i = 1, 2)$ with $N_i = v_i w_i^\top$, note first that $\operatorname{tr}(N_i) = w_i^\top v_i = 0$, so the linear terms drop:

$$\operatorname{tr}(U_1 U_2) = \operatorname{tr}\big(I + N_1 + N_2 + N_1 N_2\big) = 2 + \operatorname{tr}(N_1 N_2).$$

Moreover $N_1 N_2 = v_1 (w_1^\top v_2) w_2^\top$, where $w_1^\top v_2$ is a scalar, so by cyclicity of the trace

$$\operatorname{tr}(N_1 N_2) = (w_1^\top v_2)\ \operatorname{tr}(v_1 w_2^\top) = (w_1^\top v_2)(w_2^\top v_1), \qquad \text{whence} \qquad \operatorname{tr}(U_1 U_2) = 2 + (w_1^\top v_2)(w_2^\top v_1).$$

If the eigenlines $\mathbb{R}v_1, \mathbb{R}v_2$ are distinct, then $\{v_1, v_2\}$ is a basis; since $w_1 \neq 0$ and $w_1^\top v_1 = 0$, necessarily $w_1^\top v_2 \neq 0$, and likewise $w_2^\top v_1 \neq 0$. Hence $\operatorname{tr}(U_1 U_2) \neq 2$. As $\det(U_1 U_2) = 1$, the product cannot have 1 as an eigenvalue, contradicting the hypothesis. Therefore any two nonidentity unipotents in $\bar{\Gamma}$ share their (unique) eigenline; fixing one nonidentity $U_0 \in \bar{\Gamma}$ with eigenline ℓ, every nonidentity element of $\bar{\Gamma}$ then has eigenline ℓ. A common eigenvector spanning ℓ simultaneously triangularises $\bar{\Gamma}$.

In either case, after a change of basis $\bar{\Gamma}$ consists of upper triangular matrices, and the top-left and bottom-right entries, $M \mapsto M_{11} =: \chi(M)$ and $M \mapsto M_{22} =: \psi(M)$, are homomorphisms $\bar{\Gamma} \to \mathbb{R}^\times$ (the product of upper triangular matrices multiplies the diagonal entries entrywise). Every element $M = \left(\begin{smallmatrix} \chi(M) & * \\ 0 & \psi(M) \end{smallmatrix}\right)$ has eigenvalues $\chi(M), \psi(M)$, so the hypothesis $1 \in \operatorname{spec}(M)$ for all M reads $\bar{\Gamma} = \ker(\chi - 1) \cup \ker(\psi - 1)$, a union of two subgroups. A group is never the union of two *proper* subgroups,[2] so $\chi \equiv 1$ or $\psi \equiv 1$; that is, $\bar{\Gamma} \subseteq \mathcal{U}_1$ or $\bar{\Gamma} \subseteq \mathcal{U}_2$. $\square$

Applying Lemma 9 to $\bar{\Gamma}$ and changing affine coordinates accordingly, we are in one of two situations:

$$\textbf{Form A:} \quad \gamma(x, y) = (x + a_\gamma y + u_\gamma,\ b_\gamma y + v_\gamma), \qquad b_\gamma \neq 0; \tag{2}$$

$$\textbf{Form B:} \quad \gamma(x, y) = (\alpha_\gamma x + \beta_\gamma y + \mu_\gamma,\ y + \tau_\gamma), \qquad \alpha_\gamma \neq 0. \tag{3}$$

A linear part in $\mathcal{U}_1 = \left(\begin{smallmatrix} 1 & a \\ 0 & b \end{smallmatrix}\right)$ gives Form A (its translation part written (u_γ, v_γ)), and one in $\mathcal{U}_2 = \left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right)$ gives Form B (translation part $(\mu_\gamma, \tau_\gamma)$). We treat Form B first (Section 4); Form A is reduced to it (Section 5), where the precise passage between the two parameter sets is made where it is needed.

Remark. The normalisations below repeatedly use *form-preserving* affine coordinate changes. For Form B these are the shears $(x, y) \mapsto (x + ky, y)$, the translations, and the separate rescalings of x and of y: each conjugates $\left(\begin{smallmatrix} \alpha & \beta \\ 0 & 1 \end{smallmatrix}\right)$ into a matrix of the same shape and leaves α (hence the hyperbolic/nilpotent dichotomy below) unchanged; and analogously for Form A. We use such changes without further comment.

4 Form B: the second coordinate is translated

Recall the plan: discreteness in $\operatorname{Aff}(\mathbb{R}^2)$ - the property (1) - is already in hand, and it remains to show that in each normal form Γ lands inside a closed subgroup acting *properly*, whereupon Lemma 7 concludes. In Form B the second coordinate is a clean one-dimensional translation action, and the homomorphism $\tau \colon \Gamma \to \mathbb{R}$ introduced below splits the analysis into a *hyperbolic* subcase (some $\alpha_\gamma \neq 1$; driven by Corollary 5) and a *nilpotent* subcase (all $\alpha_\gamma = 1$; driven by a recurrence argument).

[2]If $G = A \cup B$ with $A, B < G$, pick $a \in A \setminus B$ and $b \in B \setminus A$; then ab lies in neither A nor B, since $ab \in A$ would force $b = a^{-1}(ab) \in A$, and $ab \in B$ would force $a = (ab)b^{-1} \in B$.

Assume Γ is in Form (3). The map $\gamma \mapsto \operatorname{sign}(\alpha_\gamma)$ is a homomorphism $\Gamma \to \{\pm 1\}$ (because $\alpha_{\gamma\delta} = \alpha_\gamma \alpha_\delta$); replacing Γ by its kernel, a subgroup of index at most 2, we may assume $\alpha_\gamma > 0$ for all γ. By Lemma 6 this is harmless. Set

$$\tau\colon \Gamma \to \mathbb{R}, \quad \gamma \mapsto \tau_\gamma, \qquad K := \ker \tau.$$

Since $\tau_{\gamma\delta} = \tau_\gamma + \tau_\delta$, K is a subgroup.

Lemma 10. *Every nonidentity element of K is a pure horizontal translation $T_c(x, y) = (x+c, y)$ with $c \neq 0$. Consequently, by (1), either $K = \{\mathrm{id}\}$ or $K = \langle T_c \rangle \cong \mathbb{Z}$ for some $c \neq 0$.*

Proof. For $k \in K$, $k(x, y) = (\alpha x + \beta y + \mu,\ y)$. If $\alpha \neq 1$, then for each fixed y the equation $x = \alpha x + \beta y + \mu$ has a solution, so k has a fixed point; freeness forces $k = \mathrm{id}$. If $\alpha = 1$ but $\beta \neq 0$, then k fixes the whole horizontal line $y = -\mu/\beta$, again impossible unless $k = \mathrm{id}$. Hence a nonidentity $k \in K$ has $\alpha = 1, \beta = 0$, i.e. $k = T_c$ with $c = \mu \neq 0$ (and $c \neq 0$ since $k \neq \mathrm{id}$). The set of such c is a subgroup of $\mathbb{R}$ which is discrete by (1) (the pure translations T_c form a closed line in $\mathrm{Aff}(\mathbb{R}^2)$), hence trivial or infinite cyclic. $\square$

4.1 The hyperbolic subcase: some $\alpha_\gamma \neq 1$

We normalise one such element to $h(x, y) = (\lambda x,\ y + 1)$ and then use Corollary 5 to strip every other element down to $(\alpha x,\ y + \tau)$, landing in Kuiper's hyperbolic model.

Suppose some $h \in \Gamma$ has $\alpha_h = \lambda \neq 1$. Then $h \notin K$, so $\tau_h \neq 0$ (by Lemma 10, since elements of K have $\alpha = 1$). The linear part $\left(\begin{smallmatrix} \lambda & \beta_h \\ 0 & 1 \end{smallmatrix}\right)$ has distinct eigenvalues $\lambda \neq 1$, hence is diagonalised by a shear; rescaling y makes $\tau_h = 1$, and, since $\lambda \neq 1$, the affine map $x \mapsto \lambda x + (\text{const})$ has a fixed point which we move to the origin by an x-translation. After these coordinate changes and possibly replacing h by h^{-1},

$$h(x, y) = (\lambda x,\ y + 1), \qquad \lambda > 1. \tag{4}$$

Let $\gamma(x, y) = (\alpha x + \beta y + \mu,\ y + \tau)$ be arbitrary. Applying h^n, then γ, then h^{-n} (with $h^{\pm n}(x, y) = (\lambda^{\pm n} x,\ y \pm n)$) one computes

$$h^{-n} \gamma h^n(x, y) = \big(\alpha x + \lambda^{-n} \beta y + \lambda^{-n}(\beta n + \mu),\ y + \tau\big). \tag{5}$$

As $n \to \infty$ the right-hand side converges in $\mathrm{Aff}(\mathbb{R}^2)$ to $(x, y) \mapsto (\alpha x,\ y + \tau)$. The elements (5) lie in Γ, so by Corollary 5 the sequence is eventually constant. Eventual constancy forces $\lambda^{-n}\beta$ and $\lambda^{-n}(\beta n + \mu)$ to be independent of n; as both tend to 0, we get $\beta = 0$ and then $\mu = 0$. (Taking $\gamma = T_c \in K$ here gives $\lambda^{-n} c$ eventually constant, hence $c = 0$: thus $K = \{\mathrm{id}\}$ automatically in the hyperbolic subcase.)

Therefore every element has the form $\gamma(x, y) = (\alpha_\gamma x,\ y + \tau_\gamma)$. If $\tau_\gamma = 0$ then γ fixes the line $x = 0$, so freeness gives $\gamma = \mathrm{id}$; hence τ is injective. The orbit of $(0, 0)$ is $\{(0, \tau_\gamma) : \gamma \in \Gamma\}$, which is discrete, so $\tau(\Gamma)$ is a discrete subgroup of $\mathbb{R}$, i.e. $\tau(\Gamma) = a\mathbb{Z}$ for some $a \neq 0$. Thus $\Gamma \cong \mathbb{Z}$, generated by a single element

$$g(x, y) = (\lambda_0 x,\ y + a), \qquad \lambda_0 > 0,\ \lambda_0 \neq 1$$

(the value $\lambda_0 \neq 1$ because $h = g^k$ has $\alpha_h = \lambda_0^k = \lambda \neq 1$). Then $g^n(x, y) = (\lambda_0^n x,\ y + na)$ shifts the second coordinate by na, so for compact K only finitely many n satisfy $g^n K \cap K \neq \varnothing$. Hence $\langle g \rangle$ acts properly discontinuously. This is Kuiper's hyperbolic model H.

4.2 The nilpotent subcase: $\alpha_\gamma = 1$ for all γ

Here Γ sits in the three-dimensional Heisenberg group; the one obstruction is noncommutativity, which a recurrence argument (Proposition 12) rules out, after which Γ falls into a translation or parabolic model.

Now every element is

$$\gamma(x,y) = (x + \beta_\gamma y + \mu_\gamma,\ y + \tau_\gamma), \tag{6}$$

which we identify with the unipotent matrix $\begin{pmatrix} 1 & \beta_\gamma & \mu_\gamma \\ 0 & 1 & \tau_\gamma \\ 0 & 0 & 1 \end{pmatrix}$ acting on $(x,y,1)^\mathsf{T}$. Thus Γ lies in the three-dimensional Heisenberg group N, with multiplication

$$\beta_{\gamma\delta} = \beta_\gamma + \beta_\delta, \quad \tau_{\gamma\delta} = \tau_\gamma + \tau_\delta, \quad \mu_{\gamma\delta} = \mu_\gamma + \mu_\delta + \beta_\gamma\tau_\delta. \tag{7}$$

Only the μ-coordinate is non-additive; the cross-term $\beta_\gamma\tau_\delta$ is the sole source of noncommutativity in N, and it is precisely what will surface as D below.

With the convention $[\gamma,\delta] = \gamma\delta\gamma^{-1}\delta^{-1}$ we read the commutator off (7). Since β and τ are additive homomorphisms, $[\gamma,\delta]$ has $\beta = \tau = 0$ and is therefore a pure horizontal translation T_c (the centre of N consists of the T_c); writing out its μ-coordinate from (7) gives $c = \beta_\gamma\tau_\delta - \beta_\delta\tau_\gamma$, so

$$[\gamma,\delta] = T_{D(\gamma,\delta)}, \qquad D(\gamma,\delta) := \beta_\gamma\tau_\delta - \beta_\delta\tau_\gamma. \tag{8}$$

Equivalently, $D(\gamma,\delta)$ is the determinant of the matrix with rows $(\beta_\gamma,\tau_\gamma)$ and $(\beta_\delta,\tau_\delta)$, so

$$D(\gamma,\delta) \neq 0 \quad \Longleftrightarrow \quad (\beta_\gamma,\tau_\gamma),\ (\beta_\delta,\tau_\delta) \text{ are linearly independent in } \mathbb{R}^2. \tag{9}$$

We must show Γ is abelian; the dynamics enters exactly here.

Lemma 11 (Invariant torus and recurrence)**.** *Keep the notation of (6)–(8), and suppose $K = \ker\tau = \langle T_c\rangle$ with $c \neq 0$. Let $a, b \in \Gamma$ satisfy $\tau_a \neq 0$ and $D(a,b) \in c\mathbb{Z}$, and put $\Lambda := \langle T_c, a\rangle$. Then Λ is a subgroup of Γ isomorphic to $\mathbb{Z}^2$ that acts freely, properly discontinuously and cocompactly on $\mathbb{R}^2$; and there exist a point $p \in \mathbb{R}^2$, integers $n_j \to +\infty$ and elements $h_j \in \Lambda$ such that $g_j := h_j^{-1}b^{n_j} \in \Gamma$ satisfies $g_j p \to p$.*

Proof. Since T_c is central, T_c and a commute, and $\langle T_c\rangle \cap \langle a\rangle = \{\mathrm{id}\}$ (a relation $a^n = T_c^m$ forces $n\tau_a = 0$, i.e. $n = 0$). Hence $\Lambda \cong \mathbb{Z}^2$, and as a subgroup of Γ it acts freely. It acts properly discontinuously: for a compact C, the second coordinate of $T_c^m a^n$ is shifted by $n\tau_a$ (since T_c preserves the second coordinate and a shifts it by τ_a), so the boundedness of C leaves only finitely many n with $T_c^m a^n C \cap C \neq \varnothing$; and then, the first coordinate being shifted by mc ($c \neq 0$), only finitely many m. It acts cocompactly, with fundamental region $Q := [-|c|/2,\ |c|/2] \times [-|\tau_a|/2,\ |\tau_a|/2]$: a suitable a^{-n} brings the second coordinate of any point of $\mathbb{R}^2$ into $[-|\tau_a|/2, |\tau_a|/2]$, and then a suitable T_c^{-m} brings the first into $[-|c|/2, |c|/2]$, so $\mathbb{R}^2 = \Lambda\cdot Q$ with Q compact. Hence $\mathbb{T} := \mathbb{R}^2/\Lambda$ is a compact metrizable surface (in fact a 2-torus; only its compactness and metrizability are used) and $\pi\colon \mathbb{R}^2 \to \mathbb{T}$ is a covering map.

Next, b normalises Λ: $bT_cb^{-1} = T_c$ (centrality) and, by (8), $bab^{-1} = [b,a]\,a = T_{-D(a,b)}a = T_c^{-D(a,b)/c}a \in \Lambda$ (recall $D(a,b) \in c\mathbb{Z}$), so b descends to a homeomorphism $\bar b\colon \mathbb{T} \to \mathbb{T}$. Every homeomorphism of a nonempty compact metric space has a recurrent point [2] (a nonempty closed $\bar b$-invariant minimal set exists by Zorn's lemma, and every point of a minimal set is recurrent), so there are $\bar p \in \mathbb{T}$ and integers $n_j \to +\infty$ with $\bar b^{n_j}\bar p \to \bar p$. Fix a lift $p \in \mathbb{R}^2$ of $\bar p$ and $\varepsilon > 0$ so small that π maps $B(p,\varepsilon)$ homeomorphically onto its image. For all large j, $\bar b^{n_j}\bar p \in \pi\big(B(p,\varepsilon)\big)$; since $b^{n_j}p$ is a lift of $\bar b^{n_j}\bar p$, a unique $h_j \in \Lambda$ pulls it into the ball, i.e. $h_j^{-1}b^{n_j}p \in B(p,\varepsilon)$. Set $g_j := h_j^{-1}b^{n_j} \in \Gamma$. As π is injective on $B(p,\varepsilon)$ and $\pi(g_jp) = \bar b^{n_j}\bar p \to \bar p = \pi(p)$, we get $g_jp \to p$. $\square$

Proposition 12. *Under* (6)*, freeness and* (1)*, we have* $D(\gamma,\delta)=0$ *for all* $\gamma,\delta\in\Gamma$*.*

Proof. If $K=\ker\tau=\{\mathrm{id}\}$, then $[\gamma,\delta]\in\ker\tau=\{\mathrm{id}\}$ for all γ,δ (the commutator (8) has $\tau=0$), so $D\equiv 0$ at once. Assume therefore $K=\langle T_c\rangle$ with $c\neq 0$ (Lemma 10), and suppose for contradiction that $D(a,b)\neq 0$ for some $a,b\in\Gamma$.

Reductions. By (8), $[a,b]=T_{D(a,b)}\in\ker\tau=K=\langle T_c\rangle$, so $D:=D(a,b)\in c\mathbb{Z}\setminus\{0\}$. Moreover $\tau_a\neq 0$: if $\tau_a=0$ then $a\in K$ is a pure translation, so $\beta_a=0$ and the vector (β_a,τ_a) vanishes, contradicting the linear independence (9); likewise $\tau_b\neq 0$. Thus the hypotheses of Lemma 11 hold, and it provides the lattice $\Lambda=\langle T_c,a\rangle$, a point $p\in\mathbb{R}^2$, integers $n_j\to+\infty$ and $h_j\in\Lambda$ with $g_j:=h_j^{-1}b^{n_j}\in\Gamma$ and $g_jp\to p$.

Nontriviality. For $n\geq 1$ we have $b^n\notin\Lambda$: otherwise, comparing the (β,τ)-coordinates (unaffected by the central factor T_c), $(n\beta_b,n\tau_b)=(m\beta_a,m\tau_a)$ for some $m\in\mathbb{Z}$, making (β_b,τ_b) a scalar multiple of (β_a,τ_a) and contradicting their linear independence (9). Hence each $g_j=h_j^{-1}b^{n_j}\neq\mathrm{id}$ (else $b^{n_j}=h_j\in\Lambda$ with $n_j\geq 1$). So (g_j) is a sequence in $\Gamma\setminus\{\mathrm{id}\}$ with $g_jp\to p$, contradicting Lemma 3. Hence $D\equiv 0$. $\square$

By Proposition 12 and (9), the vectors $(\beta_\gamma,\tau_\gamma)$ are pairwise linearly dependent. If all these vectors vanish, then every $\tau_\gamma=0$, so $\Gamma=K$ is a cyclic group of translations (Lemma 10), which acts properly discontinuously. Otherwise fix one nonzero vector (β_0,τ_0); vanishing of all determinants makes every $(\beta_\gamma,\tau_\gamma)$ proportional to it. If $\tau_0=0$ then every $\tau_\gamma=0$ and again $\Gamma=K$; so $\tau_0\neq 0$, and with $r:=\beta_0/\tau_0$ we get $\beta_\gamma=r\tau_\gamma$ for all γ, so

$$\Gamma\ \subseteq\ P_r:=\big\{\,(x,y)\mapsto(x+rsy+t,\ y+s):s,t\in\mathbb{R}\,\big\}.$$

Lemma 13. P_r *is a closed abelian subgroup of* $\mathrm{Aff}(\mathbb{R}^2)$ *and acts properly on* $\mathbb{R}^2$ *in the sense of Lemma 7.*

Proof. Writing $g_{s,t}(x,y)=(x+rsy+t,y+s)$, one checks $g_{s_1,t_1}g_{s_2,t_2}=g_{s_1+s_2,t_1+t_2+rs_1s_2}$, so P_r is a subgroup, and the commutator of two elements is $g_{0,r(s_1s_2-s_2s_1)}=\mathrm{id}$, so P_r is abelian; it is visibly closed (the parametrisation by $(s,t)\in\mathbb{R}^2$ is a homeomorphism onto a closed subset of $\mathrm{Aff}(\mathbb{R}^2)$). For properness, let $K\subseteq\mathbb{R}^2$ be compact and suppose $g_{s,t}K\cap K\neq\varnothing$, say $(x,y)\in K$ and $g_{s,t}(x,y)=(x+rsy+t,y+s)\in K$. Then y and $y+s$ both lie in a bounded set, so s is bounded; and x and $x+rsy+t$ are bounded with rsy bounded (since s,y are), so t is bounded. Thus $\{(s,t):g_{s,t}K\cap K\neq\varnothing\}$ is bounded; being also closed, it is compact. $\square$

Combining $\Gamma\subseteq P_r$ with (1) and Lemmas 7, 13, the group Γ acts properly discontinuously. This is Kuiper's translation/parabolic model ($r=0$ gives T, $r\neq 0$ gives P). This completes Form B.

5 Form A: reduction to Form B

The goal here is not a fresh analysis but a *reduction*: we show that every Form-A group can be brought into Form B (after passing to a subgroup of index ≤ 2), so that Section 4 applies. When some linear part is non-unipotent this reduction is the work of Steps A and B below; when all linear parts are unipotent it is immediate.

Assume now Γ is in Form (2): $\gamma(x,y)=(x+a_\gamma y+u_\gamma,\ b_\gamma y+v_\gamma)$. As in Section 4 the sign of b_γ is a homomorphism to $\{\pm1\}$ (because $b_{\gamma\delta}=b_\gamma b_\delta$); passing to $\Gamma^+:=\ker(\operatorname{sign}b_\bullet)$, of index at most 2 (harmless by Lemma 6), we may assume $b_\gamma>0$ for all γ. Now two cases arise. If $b_\gamma=1$ for every $\gamma\in\Gamma^+$, then Γ^+ is in Form B (with $\tau_\gamma=v_\gamma$, $\alpha_\gamma=1$) and acts properly discontinuously by Section 4.2; hence so does Γ, again by Lemma 6, and we are done. Otherwise choose $h\in\Gamma^+$ with $b_h\neq 1$. Its linear part $\left(\begin{smallmatrix}1&a_h\\0&b_h\end{smallmatrix}\right)$ has distinct eigenvalues 1 and b_h; diagonalising by a shear, killing the y-constant (possible as $b_h\neq 1$) and rescaling x (possible as h is fixed-point free, so its eigenvalue-1 direction acts by a nonzero translation), and replacing h by h^{-1} if necessary, we arrange

$$h(x,y) = (x+1,\ \lambda y), \qquad \lambda > 1. \tag{10}$$

Note that the coordinate roles are interchanged relative to (4): here h *translates* the first coordinate (by $+1$) and *scales* the second (by λ), the reverse of the hyperbolic subcase. This is why below the equidistribution argument will act on the first coordinate and the contraction λ^{-n} on the second.

Step A: every $v_\gamma = 0$

Steps A and B will strip an arbitrary element down to $\gamma(x,y) = (x+u_\gamma,\ b_\gamma y)$ - first forcing $v=0$ (Step A), then $a=0$ (Step B) - after which a coordinate swap lands us in the hyperbolic subcase of Form B. Step A mirrors the recurrence argument of Section 4.2, with equidistribution (Lemma 14) in the role of recurrence: we manufacture nonidentity elements g_j with $g_j(0,0) \to (0,0)$, contradicting Lemma 3. The idea is that the commutator $c = [h,\gamma]$ is a unipotent (Form-B-type) element whose *second* coordinate can be contracted by conjugating with a high power of h (which scales y by λ^{-n}), while its *first* coordinate is a quadratic polynomial $P(m)$ in the power m, which equidistribution drives to an integer - and an h-power then to 0. So: make the first coordinate $\to 0$ via the choice of m, then the second $\to 0$ via a large power of h.

Let $\gamma(x,y) = (x+ay+u,\ by+v)$ be arbitrary and suppose, for contradiction, that $v \neq 0$. Consider the commutator $c := [h,\gamma] = h\gamma h^{-1}\gamma^{-1}$.

The second coordinate transforms independently of the first in Form A, so the y-dynamics is a one-dimensional affine action: h acts by $h_y\colon y \mapsto \lambda y$ and γ by $\gamma_y\colon y \mapsto by+v$. Computing in $\mathrm{Aff}(\mathbb{R})$,

$$\gamma_y^{-1}(y) = \frac{y-v}{b} \xrightarrow{h_y^{-1}} \frac{y-v}{b\lambda} \xrightarrow{\gamma_y} \frac{y-v}{\lambda} + v \xrightarrow{h_y} y + (\lambda-1)v.$$

Hence the y-part of c is the translation $y \mapsto y+V$ with $V = (\lambda-1)v \neq 0$. The linear part of c is the group commutator

$$[L_h, L_\gamma] = \mathrm{diag}(1,\lambda)\left(\begin{smallmatrix}1 & a\\ 0 & b\end{smallmatrix}\right)\mathrm{diag}\!\left(1,\tfrac{1}{\lambda}\right)\left(\begin{smallmatrix}1 & a\\ 0 & b\end{smallmatrix}\right)^{-1} = \left(\begin{smallmatrix}1 & A\\ 0 & 1\end{smallmatrix}\right), \qquad A = \tfrac{a}{b}\left(\tfrac{1}{\lambda}-1\right),$$

which is unipotent. Thus c is a single element of the unipotent shape $c(x,y) = (x+Ay+U,\ y+V)$ with $V \neq 0$, corresponding as in (6) to the unipotent matrix

$$M := \begin{pmatrix}1 & A & U\\ 0 & 1 & V\\ 0 & 0 & 1\end{pmatrix} = I + N, \qquad N^3 = 0.$$

The binomial expansion $M^m = (I+N)^m = I + mN + \binom{m}{2}N^2$ then gives

$$c^m \longleftrightarrow M^m = \begin{pmatrix}1 & mA & mU + \binom{m}{2}AV\\ 0 & 1 & mV\\ 0 & 0 & 1\end{pmatrix},$$

so, reading off the last column (M^m applied to $(0,0,1)^\mathsf{T}$),

$$c^m(0,0) = \big(P(m),\ mV\big), \qquad P(m) = mU + \tfrac{AV}{2}\,m(m-1). \tag{11}$$

Lemma 14 (Polynomial recurrence). *There exist integers $m_j \to +\infty$ and $k_j \in \mathbb{Z}$ with $P(m_j) + k_j \to 0$, i.e. $\|P(m_j)\| \to 0$, where $\|\cdot\|$ is the distance to the nearest integer.*

Proof. Write $P(m) = c_2 m^2 + c_1 m$ with $c_2 = \frac{AV}{2}$ and $c_1 = U - \frac{AV}{2}$ (so $P(0) = 0$ and $\deg P \leq 2$). Either some c_i is irrational, or both are rational; these cases are exhaustive. If some c_i is irrational, Weyl's equidistribution theorem [7] - in its polynomial form, equidistribution holds as soon as one nonconstant coefficient is irrational - says $\big(P(m) \bmod 1\big)_{m\geq 1}$ is equidistributed in $[0, 1)$, in particular dense; so for each j there are arbitrarily large m with $\|P(m)\| < 1/j$, from which we extract an increasing sequence $m_j \to +\infty$. If both c_i are rational, let M be a common denominator of c_1, c_2; then for $m_j = jM$ both $c_2 m_j^2$ and $c_1 m_j$ are integers, so $P(m_j) \in \mathbb{Z}$ and $\|P(m_j)\| = 0$. In either case set $k_j = -\lfloor P(m_j) \rceil$ (nearest integer). $\square$

Now, using $h^n(x, y) = (x + n,\ \lambda^n y)$ and $c^m(x, y) = \big(x + (mA)y + P(m),\ y + mV\big)$ - so that at $y = 0$ the shear term drops and c^m merely adds $P(m)$ to the first coordinate - compute for integers s:

$$\begin{aligned} h^n(0, 0) &= (n, 0), \\ c^m(n, 0) &= (n + P(m),\ mV), \\ h^s\big(n + P(m),\ mV\big) &= (n + P(m) + s,\ \lambda^s mV). \end{aligned}$$

With $m = m_j$, $n = n_j$ and $s = k_j - n_j$ this gives the element $g_j := h^{k_j - n_j} c^{m_j} h^{n_j} \in \Gamma$ acting by

$$g_j(0, 0) = \big(P(m_j) + k_j,\ \lambda^{k_j - n_j} m_j V\big). \tag{12}$$

Three indices appear, each with a distinct job: m_j is the power of the commutator c, chosen via Lemma 14 so that $P(m_j)$ lies near an integer; k_j is the nearest-integer correction, realised as a power of h (which shifts the first coordinate by $+1$ per step), so that $P(m_j) + k_j \to 0$; and n_j is the power of the conjugating h^{n_j}, chosen *last* and large so that the second coordinate $\lambda^{k_j - n_j} m_j V \to 0$ (here h scales the second coordinate by $\lambda > 1$). Concretely: choose first m_j, k_j as in Lemma 14, so the first coordinate tends to 0; then, for each j, choose n_j so large that $\big|\lambda^{k_j - n_j} m_j V\big| < 1/j$ (possible since $\lambda > 1$). Crucially the first coordinate $P(m_j) + k_j$ does not involve n_j, so enlarging n_j leaves the first-coordinate limit intact. Then $g_j(0, 0) \to (0, 0)$. The second coordinate $\lambda^{k_j - n_j} m_j V$ is *nonzero* (all factors are nonzero), so $g_j(0, 0) \neq (0, 0)$ and hence $g_j \neq \mathrm{id}$. Thus (g_j) is a sequence in $\Gamma \setminus \{\mathrm{id}\}$ with $g_j(0, 0) \to (0, 0)$, contradicting Lemma 3 (at $p = (0, 0)$). Therefore $v = 0$ for every γ.

Step B: every $a_\gamma = 0$

With $v = 0$, an arbitrary element is $\gamma(x, y) = (x + ay + u,\ by)$. Using $h^{-n}(x, y) = (x - n,\ \lambda^{-n} y)$,

$$h^n \gamma h^{-n}(x, y) = (x + a\lambda^{-n} y + u,\ by) \xrightarrow[n\to\infty]{} (x + u,\ by).$$

These lie in Γ, so by Corollary 5 the sequence is eventually constant, forcing $a\lambda^{-n}$ to be independent of n, i.e. $a = 0$ (as $\lambda > 1$). Thus every element is $\gamma(x, y) = (x + u_\gamma,\ b_\gamma y)$.

Conclusion of Form A

Steps A and B have reduced every element of Γ^+ to $\gamma(x, y) = (x + u_\gamma,\ b_\gamma y)$, a pure x-translation composed with a y-scaling. Interchanging the two coordinates $(X, Y) := (y, x)$ turns this into $(X, Y) \mapsto (b_\gamma X,\ Y + u_\gamma)$, which is exactly Form B (and restores the coordinate roles of (4): second coordinate translated, first scaled). By Section 4 (the element h has $b_h = \lambda \neq 1$, placing us in the hyperbolic subcase 4.1), Γ^+ acts properly discontinuously; hence so does Γ by Lemma 6.

6 Proof of the Theorem and the quotient

Proof of Theorem 1. By Proposition 4, Γ satisfies (1). By Lemmas 8 and 9 we may, after a linear change of coordinates, assume Γ is in Form A or Form B. Form B is handled in Section 4 (hyperbolic subcase 4.1, or nilpotent subcase 4.2 where Proposition 12 places Γ in a cyclic translation group or in some P_r). Form A reduces to Form B in Section 5. In every branch Γ acts properly discontinuously, either directly (cyclic groups) or via Lemma 7 applied to the closed properly-acting group P_r (Lemma 13). Since affine changes of coordinates and passage to a finite-index subgroup preserve both hypotheses and (by Lemma 6) the conclusion, this proves Γ acts properly discontinuously.

Finally, since Γ acts freely and properly discontinuously by homeomorphisms of $\mathbb{R}^2$, it is a standard fact [4] that the quotient map $\mathbb{R}^2 \to \mathbb{R}^2/\Gamma$ is a covering and $\mathbb{R}^2/\Gamma$ is a (Hausdorff) topological surface. Because the deck transformations are affine, the charts are affine and $\mathbb{R}^2/\Gamma$ is a complete flat affine surface. □

Remark 15. The trichotomy obtained is, up to finite index and affine conjugacy, exactly Kuiper's list of plane models: a cyclic *hyperbolic* group $H = \langle (x, y) \mapsto (\lambda_0 x, y+a) \rangle$ (Section 4.1); a subgroup of the *translation* group $T = P_0$; or a subgroup of a *parabolic* group $P = P_r$, $r \neq 0$ (Section 4.2). Thus, in dimension two, the weak hypothesis "*free with discrete orbits*" already forces proper discontinuity, and the only obstruction one might fear - a discrete-but-non-closed orbit accumulating onto another orbit - is ruled out by the two recurrence arguments of Sections 4.2 and 5.

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Statement on the use of AI tools

During the preparation of this work the author used an AI-based assistant (Anthropic's Claude) to help with language and exposition, with the literature search, and with independently re-checking the computations. The mathematical results and proofs are the author's own; the author reviewed and verified all content and takes full responsibility for it.

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