On the construction of compactifications of homogenous spaces.

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1 Introduction

The thesis at hand is giving an exposition of the wonderful compactification construction of a semisimple group of adjoint type. The construction is due to C. de Concini and C. Procesi and can be found in [1]. They consider the case of a semisimple algebraic group of adjoint type G and an involution $\sigma : G \to G$. If we denote $H = G^\sigma$ the subgroup of fixed points of σ the result is as follows. Let l be the dimension of a maximal torus in H. Then there is a projective variety X with the following properties:

1) X has an open dense G orbit isomorphic to G/H.
2) X is smooth with finitely many G orbits.
3) The orbit closures are all smooth.
4) There is a 1-1 correspondence between the set of orbit closures and subsets of $I = \{1, \dots, l\}$. If $J \subset I$ we denote S_J the corresponding orbit closure.
5) It holds that $S_J \cap S_K = S_{J \cup K}$ and $codim(S_J) = |J|$.
6) Each S_J is the transversal crossings of the S_i with $i \in J$.

In this thesis we consider a special case of this construction. We take G to be a semisimple group of adjoint type and focus on the involution of $G \times G$ given by

$$\begin{aligned} \sigma : G \times G &\to G \times G \\ (g_1, g_2) &\mapsto (g_2, g_1). \end{aligned}$$

The fixed point group of σ coincides with G via the diagonal embedding

$$\begin{aligned} diag : G &\to G \times G \\ g &\mapsto (g, g) \end{aligned}$$

and the quotient $(G \times G)/diag(G)$ is again isomorphic to G. In this case we call X the wonderful compactification of G.

Furthermore the compactification X is a spherical variety for $G \times G$ that s, given a Borel subgroup $B \subset G$ we will find a open dense $B^- \times B$ orbit in X.

1.1 A simple example

Consider the case of

$$SL_2(\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} | ad - bc = 1 \right\}.$$

If we take the embedding of SL_2 into projective 3-space via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{bmatrix} a & b \\ c & d \end{bmatrix} = [a : b : c : d]$$

then the image is clearly isomorphic to $SL_2/Z(SL_2) = PSL_2$, the center being $\pm id$, and the condition on the determinant simply becomes $ad - bc \neq 0$. We can therefore identify PSL_2 with this open subset of $\mathbb{P}^3$. The Zariski closure of the image yields all of $\mathbb{P}^3$.

We want to describe the complement to PSL_2, i.e. the hypersurface given by $ad - bc = 0$ in terms of an action of PSL_2. If we just take the PSL_2 action on $\mathbb{P}^3$ that comes from left multiplication we will get a lot of orbits. If we instead take the $PSL_2 \times PSL_2$ action on $\mathbb{P}^3$ that is induced by the action

$$(g_1, g_2) \cdot g = g_1 g g_2^{-1}$$

the situation becomes much nicer. In fact we can describe the orbits explicitly: The orbit of the identity $[id]$ under this action is just PSL_2. In addition we look at the action on the vector

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \text{ explicitly: } (g_1, g_2^{-1}) \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} ae & af \\ ce & cf \end{bmatrix}$$

where we set

$$g_1 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad g_2 = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

subject to the non-vanishing of the determinant. Note that the determinant of

$$\begin{bmatrix} ae & af \\ ce & cf \end{bmatrix}$$

is vanishing and therefore this orbit is in the complement to PSL_2. A quick glance on the variables shows that this is indeed all of the complement and thereby we identified two orbits of $PSL_2 \times PSL_2$ on $\mathbb{P}_3$. One isomorphic to PSL_2 itself being open as well as dense and another one that is a hypersurface. Note that by the Euler criteria the second orbit is smooth. The choice of

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

was not coincidental. Consider the diagonal matrices in PSL_2, they are of the form

$$\begin{bmatrix} a^{-1} & 0 \\ 0 & a \end{bmatrix}$$

and note that the closure of this torus is the affine line $[1:0:x:0]$. In particular the point $[1:0:0:0]$ is in this closure. All of this is part of a more general phenomena.

While this procedure seems to be very natural for a given group we will have to take a slightly different point of view to give the construction in full generality. To be more precise: In the setting above we took the natural representation $V \simeq \mathbb{C}^2$ of SL_2 and viewed $SL_2 \subset \mathrm{End}(V)$. Finally we embedded PSL_2 into $\mathbb{P}(End(V))$ via this identification. However, in general the natural representation of a semisimple algebraic group will not have enough nice properties to provide a similar nice compactification. One of the crucial parts is to identify a representation V of a semisimple simply connected group G which has these nice properties. This is done in chapter 3 section 1. Fortunately the structure theory of semisimple algebraic groups and their representations is a very well developed area and we will obtain the necessary tools from there.

Consequently we will give the necessary background on Lie algebra's and algebraic groups and their representations in chapter two. The construction of the compactification is carried out in chapter three.

2 Background on Lie algebras, algebraic groups and their representations

Throughout this thesis we will denote $k = \mathbb{C}$ for the complex numbers.

2.1 Preliminary remark

During this chapter we will state some facts about Lie algebras and algebraic groups but don't give full proofs. Most of it are standard facts that can be found in any book about the classification of semisimple Lie algebras or linear algebraic groups.

To see how the theory unfolds we will have some working examples. While we give a little more rigorous definition later we will already introduce the examples here: our main example will be the linear group SL_n of $n \times n$ matrices with determinant equal to 1. Equally important is it's Lie algebra the $n \times n$ matrices with trace zero which we will denote with $\mathfrak{sl}_n$. More concrete we will work with SL_2 as above and use SL_3 as an example to see a little bit further when the SL_2 example is too simple.

2.2 Basics about Lie algebras

We will introduce the very basics about Lie algebras in an remark which serves as a reminder and to fix notation.

Remark 1.

1) Recall that an abstract Lie algebra is a vector space V together with a skew-symmetric bilinear map $[,] : V \times V \to V$ called the Lie bracket obeying the Jacobi identity:
$$[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0.$$
As a basic example we have the space of $n \times n$ matrices $M_n(k)$ with Lie bracket $[A, B] := A \cdot B - B \cdot A$. We will denote $M_n(k)$ with $\mathfrak{gl}_n = \mathfrak{gl}_n(k)$, because later on we will view $M_n(k)$ as the Lie algebra associated to the group $GL_n(k)$. We will also use the notation $\mathfrak{gl}(V)$ for a vector space V, respectively $\mathfrak{gl}_n(V)$ if we want to indicate the dimension of V, to denote the vector space of linear maps with Lie bracket $A \circ B - B \circ A$. Note that this is isomorphic to $\mathfrak{gl}_n$ after a choice of basis for V. Furthermore we will mostly use the german $\mathfrak{g}$ to denote a Lie algebra.

2) A morphism of Lie algebras $\rho : \mathfrak{g} \to \mathfrak{g}'$ is a linear map, which respects the Lie brackets i.e. $\rho([A, B]) = [\rho(A), \rho(B)]$. A representation of a lie algebra $\mathfrak{g}$ is a morphism of Lie algebras $\rho : \mathfrak{g} \to \mathfrak{gl}_n(V)$ for some complex vector space V. We will often just refer to a representation as V. Such a representation is called irreducible if the only subspaces $W \subset V$ that satisfy $\rho(W) \subset W$ are $W = 0$ or $W = V$.
3) One of the most important examples of a representation of $\mathfrak{g}$ is the adjoint representation which arises from the Lie bracket. Here we use that for a fixed $x \in \mathfrak{g}$ the map $[x, \cdot]$ is linear in the second argument. We therefore obtain the representation

$$\begin{aligned} ad : \mathfrak{g} &\to \mathfrak{gl}(\mathfrak{g}) \\ x &\mapsto [x, \cdot]. \end{aligned}$$

A simple utilization of the Jacobi identity shows that this is indeed a representation, i.e. it respects the Lie bracket. This representation gives rise to a bilinear form on $\mathfrak{g}$ called the Killing form via $(x, y) := K(x, y) := tr(ad(x) \cdot ad(y))$ where tr is just the ordinary trace on $\mathfrak{gl}(\mathfrak{g})$.
4) A Lie algebra $\mathfrak{g}$ is called semisimple if the Killing form $K(x, y) = tr([x, -][y, -])$ is non-degenerate. There are several characterizations of semisimplicity of a Lie algebra but we will stick to this one for this thesis.
5) Let $\mathfrak{g}$ be a semisimple Lie algebra. An element $x \in \mathfrak{g}$ is called semisimple if the linear operator

$$ad(x) : \mathfrak{g} \to \mathfrak{g}$$

is diagonalizable and x is called nilpotent if $ad(x)^n = 0$ for some $n \in \mathbb{N}$.

Example 2.2.1. Our most important example, which is in a certain way the prototype of a semisimple Lie algebra, will be $\mathfrak{sl}_2 = \mathfrak{sl}_2(k) \subset \mathfrak{gl}_2(k)$ the space of matrices with trace zero. A basis of this space is given by

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The Lie bracket here is just the matrix commutator, see Remark 1. A quick calculation shows that x, y, h satisfy the following commutator relations:

$$[x, y] = h, [h, x] = 2x, [h, y] = 2y.$$

Furthermore $\mathfrak{sl}_2(k)$ is completely determined by those relations, that is any 3-dimensional Lie algebra with a basis that enjoys the same commutator relations is actually isomorphic to $\mathfrak{sl}_2(k)$. Ordering the basis in the form x, h, y one can use the relations to read of the adjoint representation:

$$\operatorname{ad} x = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \operatorname{ad} h = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \quad \operatorname{ad} y = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}$$

and from here it is a straightforward computation to check that $\mathfrak{sl}_2$ is indeed semisimple. Furthermore we see that h is semisimple, while x, y are nilpotent.
One can also show that indeed $\mathfrak{sl}_n$ is semisimple. Given our characterization of semisimplicity this would become a quite extensive computation. There are other characterizations of semisimplicity which would be more useful in this case.

2.3 Basics about algebraic groups

We will assume that the reader is familiar with algebraic varieties. In particular we will use any topological notions freely (open, closed, irreducible, connected, dimension etc.) as well as the notion of smoothness. We will take the point of view of classical algebraic geometry, which means we will threat a variety as a quasi-projective variety defined as the solution set of a system of polynomials. This has the advantage that we can write down maps pointwise. For the basics on the language of varieties we refer to [3].

Definition 2.3.1. An *algebraic group* is a variety which is also a group, i.e. we have regular maps $m : G \times G \to G$ for multiplication, $i : G \to G$ for inversion and a neutral element $e : \operatorname{Spec}(k) \to G$ satisfying the usual group axioms:

$$\begin{array}{ccc} G \times G \times G & \xrightarrow{id \times m} & G \times G \\ {\scriptstyle m \times id}\downarrow & & \downarrow{\scriptstyle m} \\ G \times G & \xrightarrow{\quad m \quad} & G \end{array} \qquad \begin{array}{ccc} G & \xrightarrow{id \times e} & G \times G \\ {\scriptstyle e \times id}\downarrow & \searrow^{id} & \downarrow{\scriptstyle m} \\ G \times G & \xrightarrow{\quad m \quad} & G \end{array}$$

$$\begin{array}{ccc} G \times G & \xrightarrow{i \times id} & G \times G \\ {\scriptstyle \Delta}\uparrow & & \downarrow{\scriptstyle m} \\ G & \xrightarrow{\quad \epsilon \quad} & G \\ {\scriptstyle \Delta}\downarrow & & \uparrow{\scriptstyle m} \\ G \times G & \xrightarrow{id \times i} & G \times G \end{array}$$

Here Δ means the diagonal map $\Delta(g) = (g, g)$ and ϵ is the map sending everything to $e(\operatorname{Spec}(k))$.
We will denote $m(g, h) =: g \cdot h$ and $i(g) =: g^{-1}$ and $e(\operatorname{Spec}(k)) =: e$.
A morphism of algebraic groups $\varphi : G \to G'$ is a morphism of varieties which respects multiplication and sends the neutral element of G to the neutral element of G'.

Example 2.3.1. Recall that giving an affine variety is the same as giving its coordinate ring. When referring to an affine algebraic group G we will denote the coordinate ring with $k[G]$. Notice that giving the mentioned group morphisms is equivalent to giving k-algebra morphisms which we will denote as above, i.e. we have the comultiplication $m : k[G] \to k[G] \otimes k[G]$, the coinversion or antipode $i : k[G] \to k[G]$ and the identity element $e : k[G] \to k$.
This enables us to construct the invertible $n \times n$ matrices $GL_n(k)$ as an algebraic group with coordinate ring

$$k[GL_n(k)] := k[x_{11}, \ldots, x_{nn}]_{(det)}$$

where det denotes the polynomial given by the determinant. We have the morphisms

$$\begin{aligned} m : k[x_{11}, \ldots, x_{nn}]_{(det)} &\to k[x_{11}, \ldots, x_{nn}]_{(det)} \otimes k[x_{11}, \ldots, x_{nn}]_{(det)} \\ x_{ij} &\mapsto \sum_{k=1}^{n} x_{ik} \otimes x_{kj} \\ i : k[x_{11}, \ldots, x_{nn}]_{(det)} &\to k[x_{11}, \ldots, x_{nn}]_{(det)} \\ x_{ij} &\mapsto \left((x_{kl})^{-1}\right)_{ij} \\ e : k[x_{11}, \ldots, x_{nn}]_{(det)} &\to k \\ x_{ij} &\mapsto \delta_{ij} \end{aligned}$$

where $\left((x_{kl})^{-1}\right)_{ij}$ denotes the ij entry of the inverse matrix of $(x_{kl})_{kl}$ which is a polynomial via the formula $A^{-1} = det(A)^{-1}Adj(A)$. One can check that this indeed defines the usual group structure on $GL_n(k)$ making it an algebraic group. An important closed subgroup is the special linear group $SL_n(k)$ with coordinate ring

$$k[SL_n(k)] = k[GL_n(k)]/(det - 1)$$

and the same morphisms as above.

Remark 2.

1) A linear algebraic group is an algebraic group which is an affine variety. The name is not coincidental: one can proof that every linear algebraic group can be viewed as an algebraic subgroup of $GL_n(k)$ for some $n \in \mathbb{N}$, see for example [4, p. 333, Theorem 22.1.5].
2) A representation of an algebraic group G is a vector space V together with a group homomorphism
$$\rho : G \to GL(V)$$
where $GL(V) \simeq GL_n(k)$ after a choice of basis. As in the context of Lie algebras, see Remark 1, we call a representation irreducible if the only G stable subspaces,

meaning subspaces $W \subset V$ such that $\rho(G)(W) \subset W$, of V are 0 and V itself.

Definition 2.3.2.
- Let G be an algebraic group. A G-space or G-variety is a variety X over k together with a morphism of varieties

$$a : G \times X \to X$$

having the properties

1) $a(g \cdot h, x) = a(g, a(h, x))$
2) $a(e, x) = x$

for all $g, h \in G$ and all $x \in X$. We will denote $a(g, x) =: g \cdot x$. Given a G variety X with the data above we will often only say that G acts on X and write $G \curvearrowright X$. The morphism a is then just called the action of G on X.

- A G-variety X is called *homogenous* if G acts transitively on X, which means that for every $x, y \in X$ there is a $g \in G$ with $g \cdot x = y$.

- A morphism of G-spaces $\phi : X \to Y$ is a morphism of varieties that is G equivariant i.e. ϕ satisfies the equation $\phi(g \cdot x) = g \cdot \phi(x)$ for all $g \in G$ and $x \in X$.

Remark 3. Given an action of G on X and fixing $x \in X$ we can form the sets

$$G \cdot x := \{g \cdot x | g \in G\}$$

called the *orbit* of x and

$$G_x := \{g \in G | g \cdot x = x\}$$

called the *stabilizer* of x. The latter can also be formed for a subset $Z \subset X$:

$$G_Z := \{g \in G | g \cdot z \in Z \text{ for all } z \in Z\}.$$

Note that the stabilizer is always a subgroup. For an algebraic subgroup $H \subset G$ we can form the subgroups

$$\begin{aligned} N_G(H) &:= \{g \in G | g \cdot H = H \cdot g\} \\ C_G(H) &:= \{g \in G | g \cdot h = h \cdot g \text{ for all } h \in Z\} \end{aligned}$$

called the *normalizer* and the *centralizer*. The centralizer $C_G(G)$ of G itself is called the center of G and denoted with $Z(G)$. All of the objects above carry a variety structure and thus the groups in discussion are algebraic subgroups. See [5, Chapter I+II] for a reference.

We have the following basic and important result about orbits.

Theorem 2.3.1 (Closed orbit lemma). *Let G be an algebraic group acting on a non-empty variety X. Then each orbit is a smooth variety which is open in its closure in X. Its boundary is a union of orbits of strictly smaller dimension. In particular, orbits of minimal dimension are closed and closed orbits exist.*

Proof. We will only prove the second part here, for a proof of the smoothness and variety structure see [5, p. 53, Chapter 1.8].

Let $\sigma : G \times X \to X$ be the action morphism. The orbit of $x \in X$ is the image of $\sigma_x := \sigma(-, x)$ and hence by a theorem of Chevalley (See [3, II Exercise 3.19]) it is constructible, meaning there is a dense open subset $U \subset G \cdot x \subset \overline{G \cdot x}$. Since G acts transitively on $G \cdot x$ every element of $G \cdot x$ lies in a translate of U which means that $G \cdot x = \cup g \cdot U$ is open in it's closure. But that means that $\overline{G \cdot x} \setminus G \cdot x$ being the complement of a dense open set is closed and of strictly smaller dimension. Since it is also G invariant it is a union of orbits of strictly smaller dimension. This implies that orbits of smallest dimension are closed and do exist. □

Example 2.3.2. An important example of a G variety is given by a representation V of G. Recall that we have the data $\rho : G \to GL(V)$ and G acts on V via $g \cdot v = \rho(g)(v)$. Another important example is given in Remark 4.

Remark 4. It is worth noting that the quotient of a linear algebraic group G by a closed linear algebraic subgroup $H \subset G$ always exists as a quasi-projective variety. In the case that H is a normal subgroup the quotient is again a linear algebraic group. Since the main ideas of the construction will be of use later we will outline it here very roughly:

The first step is to find a representation V of G together with a line $L = k \cdot v \in V$ such that $H = G_L$. We can now take the projectivization $\mathbb{P}(V) := V \setminus \{0\}/\sim$. The action of G on V descents to an action on $\mathbb{P}(V)$ since it is linear. We now take $x = \overline{v}$ and $G/H := G \cdot x$. For details see for example [6, Chapter 5.5]. From this construction it is immediate that G/H is quasi-projective and furthermore G/H is a homogeneous G space.

The next theorem summarizes some topological properties of a connected group G and a homogeneous spaces over G.

Theorem 2.3.2. *Let G be a connected linear algebraic group. Then G is irreducible as a topological space and in particular any homogeneous G space X is irreducible.*

Proof. The second assertion is a consequence of the first since images of irreducible sets are irreducible and X is the only G orbit. For the first claim see [6]. □

2.4 The Lie algebra of a linear algebraic group

Definition 2.4.1. We define the Lie algebra of a linear algebraic group G as the tangent space $T_eG = (\mathfrak{m}_e/\mathfrak{m}_e^2)^*$. As above we will denote the Lie algebra of G with the german $\mathfrak{g}$ or sometimes with $Lie(G)$.

There are different realizations of $\mathfrak{g}$ each useful in different contexts and we will briefly explain two of them here. Recall that a derivation D of a B-algebra A is a map

$$D : A \to A$$

satisfying the Leibniz rule, i.e. $D(aa') = aD(a') + a'D(a)$ and $D(b) = 0$ for all $b \in B$. The space of all such derivations is denoted with $Der_B(A, A)$. For an algebraic group G we have an action, called left-translation, of G on $k[G]$ via

$$\lambda_x f(y) = f(x^{-1}y)$$

for all $f \in k[G]$ and $x, y \in G$. We define the space of left invariant derivations on $k[G]$ via

$$Der_k(k[G], k[G])^L := \{D \in Der_k(k[G], k[G]) : \lambda_x \circ D = D \circ \lambda_x \text{ for all } x \in G\}$$

We then have an isomorphism of vector spaces

$$T_eG \simeq Der_k(k[G], k[G])^L$$

which we explain below. The Lie bracket is now most easily defined on the left invariant derivations via

$$[D, D'] := D \circ D' - D' \circ D.$$

$Lie(G)$ behaves functorial (hence the notation) i.e. for a morphism of algebraic groups $\varphi : G \to G'$ we have a morphism of k-algebras $\varphi^* : k[G'] \to k[G]$ and therefore a map

$$\begin{aligned} d\varphi_e : Der_k(k[G], k[G])^L &\to Der_k(k[G'], k[G'])^L \\ D &\mapsto D \circ \varphi^*. \end{aligned}$$

The notation $d\varphi_e$ comes from the identification of $Der_k(k[G], k[G])^L$ with T_eG and $d\varphi_e$ can be viewed as the differential of φ at e. We will usually drop the e and just write $d\varphi$ since the basepoint is fixed in this context.

We actually have a additional isomorphism of the left invariant derivations with the point derivatives. And we will briefly explain the chain of isomorphisms $Der_k(k[G], k[G])^L \simeq Der_k(k[G], k) \simeq T_eG$ here: for the first isomorphism we have a map

$$
\begin{aligned}
Der_k(k[G],k[G])^L &\rightarrow Der_k(k[G],k) \\
D &\mapsto (f \mapsto Df(e)).
\end{aligned}
$$

And conversely given a left invariant derivation d we define the convolution of a $f \in k[G]$ with d to be

$$f * d(x) = d(\lambda_{x^{-1}} f)$$

. Then one checks that the map

$$f \mapsto f * d$$

defines a derivation which is indeed left invariant. For the second isomorphism, given a point derivation $D \in Der_k(k[G], k)$ we first observe that we can extend D to the local ring $\mathcal{O}_{G,e}$ by setting

$$D(f/g) = \frac{D(f)g - fD(g)}{g^2}.$$

Now one can check that the extended D is completely determined by it's values on $\mathfrak{m}_x$ and vanishes on $\mathfrak{m}_x^2$ we thus get a map

$$
\begin{aligned}
\delta : \mathfrak{m}_e/\mathfrak{m}_e^2 &\rightarrow k \\
\overline{f} \mapsto Df
\end{aligned}
$$

which is indeed k-linear and hence $\delta \in (\mathfrak{m}_e/\mathfrak{m}_e^2)^*$. Conversely a functional $\delta \in (\mathfrak{m}_e/\mathfrak{m}_e^2)^*$ defines a derivation by setting

$$D(f) := \delta(f - f(x) \mod (\mathfrak{m}^2))$$

.

Remark 5. Notice that a linear algebraic group is a closed subgroup of some GL_n and as such it is a Lie group in the euclidean topology. One can take this point of view to define the Lie algebra of G which is indeed done for example in [7]. From this point of view everything can be carried over from the context of complex Lie groups.

Example 2.4.1. We continue with Example 2.3.1 of GL_n and SL_n. We first look at the case of GL_n. One can check that a basis of the left invariant derivations is given by

$$f \mapsto \frac{\partial f}{\partial x_{ij}}(e)$$

where we think of the left invariant derivations as point derivations $Der_k(k[G], k[G])^L \simeq Der_k(k[G], k)$. This makes $\mathfrak{gl}_n$ isomorphic to $\mathbb{A}^{n^2} \simeq M_n$ as a vector space and one can check that the Lie bracket is indeed given by $[X, Y] = X \cdot Y - Y \cdot X$ where $\cdot$ just means the usual matrix multiplication.

Similarly one can compute that the Lie algebra of SL_n is $\mathfrak{sl}_n \subset \mathfrak{gl}_n$, the space of $n \times n$ matrices with trace zero. This can be seen most easily from the point of view of Lie Groups, see remark 5. Take a curve $A(t)$ in SL_n with $A(0) = Id$ and use the well known formula

$$grad(det(C)) \cdot D = trace(Ad(C)D)$$

where we regard D as a vector and $\cdot$ is just the usual scalar product and $Ad(C)$ is the adjoint of C. This gives

$$\begin{aligned} 0 &= \frac{d}{dt}(det(A(t)))_{|t=0} \\ &= grad(det(A(0))) \cdot A'(0) \\ &= trace(Ad(A(0))A'(0)) = trace(A'(0)). \end{aligned}$$

Where the last equality holds since $A(0) = Id$ and therefore $Ad(A(0)) = Id$ too. The Lie bracket of $\mathfrak{sl}_n$ is the same as for $\mathfrak{gl}_n$.

We have the following connection between subgroups and Lie subalgebras.

Theorem 2.4.1. *Let H, K be algebraic subgroups of a connected algebraic group G. Let $\mathfrak{h}, \mathfrak{k}$ be their Lie algebras. Then it holds that:*

1) *If $H \subset K$, then $\mathfrak{h} \subset \mathfrak{k}$.*
2) *Assume that H and K are connected. Then if $\mathfrak{h} \subset \mathfrak{k}$, then $H \subset K$.*
3) *We have $Lie(H \cap K) = \mathfrak{h} \cap \mathfrak{k}$.*

Proof. This is intuitively clear, since the tangent spaces behave well with respect to these operations, at least for smooth varieties which we have in this context. For a full proof see [4, p. 370, 24.3.5]. □

2.5 Special subgroups and their Lie algebras

Definition 2.5.1 (Special groups and Subgroups)**.**

1) Let G be a connected linear algebraic group. G is called semisimple if its Lie algebra is semisimple, see Remark 1.
2) Let G be an algebraic group and let T be a subgroup. We will denote the multiplicative group of k with $G_m := k^* := GL_1(k)$. We call T a *torus* if it is isomorphic to $(G_m)^n$ for some $n \in \mathbb{N}$. A torus is called maximal if it is maximal with respect to inclusion and being a torus.

3) Let $B \subset G$ be a subgroup. We first introduce the group commutator as $[x, y] = x \cdot y \cdot x^{-1} \cdot y^{-1}$. Now we define $B^0 = B$ and $B^n = [B^{n-1}, B^{n-1}]$. We call B *solvable* if the derived series

$$B \rhd B^1 \rhd B^2 \rhd \dots$$

eventually becomes trivial. G^{i-1} is normal in G^i and the quotient G^{i-1}/G^i is abelian. The subgroup B is called a *Borel subgroup* if it is a maximal closed connected and solvable subgroup.
4) Let $P \subset G$ be a closed subgroup. P is called a *parabolic subgroup* if the quotient G/P is a projective variety.
5) Let $G \subset GL(V)$ be a linear algebraic group. A subgroup $U \subset G$ is called *unipotent* if it is a contained in the set

$$\mathbb{U} \subset \{g \in GL(V) | (1 - g)^n = 0\}.$$

Example 2.5.1. We have already seen above that $\mathfrak{sl}_2$ is semisimple and hence SL_2 is a semisimple linear algebraic group. More generally as stated in example 2.2.1 we have that $\mathfrak{sl}_n$ is semisimple and hence SL_n is semisimple too.

Next to $(G_m)^n$ our most important example of a torus are the invertible diagonal matrices $T \subset GL_n(k)$ respectively the invertible diagonal matrices with determinant 1 in $SL_n(k)$.

One can check that the group of invertible upper triangular matrices $B \subset GL_n(k)$ is a Borel subgroup. The argument is that since an upper triangular matrix x has non-zero diagonal elements x^{-1} has as diagonal entries the inverses and hence every element of $[B, B] = B^1$ has only one's on the diagonal. An inductive argument shows that B^k has zeros on the $k-1$ upper diagonal rows. The quotient B^{k-1}/B^k is isomorphic to k^{n-k-1} and hence the claim follows.

Of course the same is true for invertible upper triangular matrices with determinant 1 inside $SL_n(k)$. Note that the torus T from above is contained in B which is not coincidental. Since T is a connected closed and solvable subgroup it has to be contained in a Borel subgroup.

There is another connection that we want to flash out. Let T and B be in GL_n as above. Notice that for any element $b \in B$ we have a decomposition of b into a diagonal and unipotent part in the following way: if $t \in T$ is the matrix that has the same diagonal elements as b we obtain that $b = t \cdot t^{-1} \cdot b$ and $t^{-1} \cdot b$ is an element in B that has only ones on the diagonal and is thus unipotent. Hence we have

$$B = TU$$

where $U \subset B$ is the subgroup of unipotent elements, i.e. the upper triangular matrices with ones on the diagonal.

In addition one can show that $GL_n(k)/B$ is a projective variety called the flag variety with respect to B. Thus B is indeed a parabolic subgroup.

The above connections hold more generally and them as well as some more properties are summarized in the following theorem.

Theorem 2.5.1. *Let G be a connected semisimple algebraic group. Then the following holds.*

1) *There exist non trivial maximal tori $T \subset G$.*
2) *All maximal tori of G are conjugate to each other, i.e. for T, T' maximal tori there exists a $x \in G$ s.t. $T' = xTx^{-1}$.*
3) *Every maximal torus T is contained in a Borel subgroup B and all Borel subgroups are conjugate.*
4) *If B is a Borel subgroup and T a torus contained in B and furthermore $U \subset B$ is the subgroup of unipotent elements of B, then $B = TU$.*
5) *Every Borel subgroup B is a parabolic subgroup.*
6) *A subgroup P is parabolic iff it contains a Borel subgroup B. This, together with the maximality of Borel subgroups with respect to being closed, connected and solvable means that the Borel subgroups are the minimal parabolic subgroups.*
7) *All subgroups of the above type are connected subgroups of G whenever G is connected.*

Proof. See [5]. □

We now make similar definitions for Lie algebras and state the connections.

Definition 2.5.2 (Subalgebras)**.**

1) Let $\mathfrak{g}$ be a semisimple Lie algebra. A subalgebra $\mathfrak{t} \subset \mathfrak{g}$ is called toral if it consists of semisimple elements and is abelian, i.e. $[x, y] = 0$ for all $x, y \in \mathfrak{t}$. A subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is called a Cartan subalgebra if it is toral and coincides with its centralizer, i.e.
$$\mathfrak{h} = \{x \in \mathfrak{g} | [x, \mathfrak{h}] = 0\}.$$
2) Let $\mathfrak{g}$ be a Lie algebra and define $D^0\mathfrak{g} := \mathfrak{g}$ and $D^n\mathfrak{g} := [D^{n-1}\mathfrak{g}, D^{n-1}\mathfrak{g}]$. We have that
$$\mathfrak{g} = D^0\mathfrak{g} \geq D^1\mathfrak{g} \geq \ldots \geq D^n\mathfrak{g} \geq \ldots$$
as subalgebras. We say that $\mathfrak{g}$ is solvable if $D^n\mathfrak{g}$ eventually becomes zero for some n.

   Now let $\mathfrak{g}$ be a semisimple Lie algebra and $\mathfrak{b} \subset \mathfrak{g}$ be a subalgebra. We call $\mathfrak{b}$ a Borel subalgebra if it is a maximal solvable subalgebra.

3) A subalgebra $\mathfrak{n} \subset \mathfrak{g}$ is called nilpotent if there is a $n \in \mathbb{N}$ such that

$$ad(x)^n = 0 \text{ for all } x \in \mathfrak{n}.$$

In other words, if $ad(x)$ is a nilpotent operator on $\mathfrak{n}$ for every $x \in \mathfrak{n}$.

Example 2.5.2. We look at the case of SL_n and it's Lie algebra $\mathfrak{sl}_n$ again. A direct computation as in the above examples shows that the Lie algebra of the diagonal matrices $D \subset SL_n$ is the set of diagonal matrices in M_n with trace zero, we call it $\mathfrak{h}$. The Lie algebra of the upper triangular matrices $B \subset SL_n$ consists of the set $\mathfrak{b} = \{b \subset M_n | b \text{ is upper triangular with trace zero}\}$ and likewise the unipotent subgroup $U \subset B$ has the strict upper triangular matrices $\mathfrak{u}$ as it's Lie algebra.

One can check that $\mathfrak{h}$ indeed coincides with it's centralizer: if x is an element in $\mathfrak{sl}_n$ that has a nonzero entry at say the ij-th position then the diagonal matrix with a one at the i-th diagonal position and a -1 at the j-th diagonal position has a non-zero commutator with x.
Now for elements $x, x' \in \mathfrak{b}$ we can decompose them into a strict upper triangular and diagonal part $x = d + u, x' = d' + u'$ and we get

$$\begin{aligned} [x, x'] &= (d+u)(d'+u') - (d'+u')(d+u) \\ &= uu' - u'u \end{aligned}$$

but the product of two strict upper triangular matrices shifts the zeros one upper diagonal higher and hence eventually becomes zero after at most n steps.

The same argument can be made for the strict upper triangular matrices $\mathfrak{u}$ proofing that these form a nilpotent Lie algebra.
Again non of these connections between the subgroups and their Lie algebras are coincidental and are summarized in the following theorem.

Theorem 2.5.2. *Let G be a connected semisimple algebraic group and $\mathfrak{g}$ its Lie algebra. Then it holds that*

1) *A Lie subalgebra of $\mathfrak{g}$ is a Cartan subalgebra if and only if it is the Lie algebra of a maximal torus of G.*
2) *A Lie subalgebra of $\mathfrak{g}$ is a Borel subalgebra if and only if it is the Lie algebra of a Borel subgroup of G.*

Proof. See [4, p. 370, 24.3.5]. □

2.6 Root systems, characters and some representation theory

Remark 6. In this section we will provide some of the structure theory of root systems and their connection with the representation theory of semisimple linear algebraic groups. It is somewhat a tough choice how to present this material. There is the possibility to give a illustrating example first and then to present the theory in greater generality or give the theory first and then give examples. Here we will follow the second way, the example will be the classification of irreducible representations of $\mathfrak{sl}_3$ which we will split up in a few parts and hopefully illustrate the theory in a convenient way.

We recall the Jordan-Chevalley decomposition for semisimple groups, which is just the familiar Jordan decomposition in a broader context.

Theorem 2.6.1. *Let $G \subset GL(V)$ be an algebraic group. An element g is called semisimple if it is diagonalizable and unipotent if $(g - id)^n = 0$ for some $n \in \mathbb{N}$. The following holds:*

1) *Every element $g \in G$ can be written in the form $g = g_s g_u$ with g_s semisimple and g_u unipotent and $g_s, g_u \in G$.*
2) *The elements g_u, g_s commute with each other.*
3) *For any morphism of linear algebraic groups $\varphi : G \to G'$ it holds that*

$$\varphi(g_s) = \varphi(g)_s \qquad \varphi(g_u) = \varphi(g)_u.$$

   In particular, if g is semisimple respective unipotent then so is $\varphi(g)$.

The analogous statements for the Lie algebra of G hold when we replace unipotent by nilpotent and multiplication by addition. Statement 3) becomes $[x_s, x_n] = 0$.

Proof. See [5, Chapter 4] □

Remark 7. Recall that a representation of a linear algebraic group G is a vector space V together with a morphism

$$\rho : G \to GL(V)$$

and by Remark 2.4.1 we have a induced representation of Lie algebras

$$d\rho : \mathfrak{g} \to \mathfrak{gl}(V).$$

We say that $W \subset V$ is a subrepresentation of V if $G \cdot W \subset W$ respectively if $\mathfrak{g} \cdot W \subset W$. The next theorem gives a useful connection between representations of G and $\mathfrak{g}$.

Theorem 2.6.2. *Let G be connected linear algebraic group and V a finite dimensional representation of G. It holds that $W \subset V$ is a subrepresentation of G if and only if it is a subrepresentation of $\mathfrak{g}$.*

Proof. This is an important result in representation theory of algebraic groups and their Lie algebras since it allows to connect the theory of irreducible representations of one another. One direction is immediate, since if we have a subrepresentation $W \subset V$ of G we can just restrict the representation to get a representation $G \to GL(W)$ and therefore a representation $\mathfrak{g} \to \mathfrak{gl}(W)$. For the other direction see [4, p. 369, 24.3.3]. □

Remark 8 (Weight spaces)**.** Let G be a connected semisimple algebraic group and T a maximal torus. By Theorem 2.5.1 such a non-trivial torus exists. Denote with $\mathfrak{t}$ its Lie algebra, by Theorem 2.5.2 this is a Cartan subalgebra. The set

$$X(T) := Hom_{Group}(T, G_m)$$

carries the structure of a group via point wise multiplication and is called the *character group* of T. Its elements are called characters and are usually denoted with a χ. One can check that $X(G_m) \simeq \mathbb{Z}$ via

$$(t \mapsto t^n) \mapsto n.$$

For $T \simeq (G_m)^n$ we have $\chi(t_1, \ldots, t_n) = \chi(t_1, 1, \ldots, 1, 1) \cdot \ldots \cdot \chi(1, 1, \ldots, 1, t_n)$ and therefore χ is determined by how it behaves on every component. We get an isomorphism $X(T) \simeq \mathbb{Z}^n$ via

$$\Big((t_1, \ldots, t_n) \mapsto t_1^{k_1} \cdot \ldots \cdot t_n^{k_n}\Big) \mapsto (k_1, \ldots, k_n).$$

We will sometimes write t^χ for $\chi(t)$ since this notation reflects the isomorphism in a natural way. For example we have $t^{\chi_1+\chi_2} = t^{\chi_1}t^{\chi_2}$.

The elements of a torus are semisimple and commute, hence by the Jordan-Chevalley decomposition Theorem 2.6.1 we have that $\rho(T)$ is simultaneously diagonalizable for a representation $\rho : G \to GL(V)$. Thus we have

$$V = \bigoplus_{\lambda \in X(T)} V_\lambda$$

where

$$V_\lambda = \{v \in V | \rho(t)v = \lambda(t)v \text{ for all } t \in T\}.$$

Now let $d\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ be the induced representation. Let $\mathfrak{t}$ be the Lie algebra of T and $\mathfrak{t}^*$ be its dual space. Since $\mathfrak{t}$ is a Cartan subalgebra, it is also simultaneously diagonalizable and thus we have a similar decomposition of V into

$$V = \bigoplus_{\lambda \in \mathfrak{t}^*} V_\lambda$$

where

$$V_\lambda = \{v \in V | \rho(t)v = \lambda(t)v \text{ for all } t \in \mathfrak{t}\}.$$

In both cases we call V_λ the weight space of V to weight λ.

Definition 2.6.1. Let G be an algebraic group. Denote the coordinate ring with $k[G]$ and for an element $x \in G$ we denote the right translation on $k[G]$ with $(\rho_x f)(y) = f(yx)$. We have an action of G on $Der_k(k[G], k[G])$ by

$$D \mapsto \rho_x \circ D \circ \rho_{x^{-1}} =: Ad(x)(D).$$

It can be checked that this restricts to an action on $\mathfrak{g} = Der_k(k[G], k[G])^L$ by using $\lambda_y \circ \rho_x = \rho_x \circ \lambda_y$ i.e. left and right translation commute. This action is indeed linear and invertible thus we obtain a representation

$$\begin{aligned} Ad : G &\rightarrow GL(\mathfrak{g}) \\ g &\mapsto Ad(g). \end{aligned}$$

Theorem 2.6.3. *Let G be a linear algebraic group and $\mathfrak{g}$ its Lie algebra. Recall that we can view $G \subset GL_n$ and therefore $\mathfrak{g} \subset \mathfrak{gl}_n$. It holds that*

1) *$Ad(g)(x) = g \cdot x \cdot g^{-1}$ where $\cdot$ is again just usual matrix multiplication.*
2) *The induced representation $d(Ad) : \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g})$ coincides with ad as defined in Remark 1 i.e. $d(Ad)(x) = [x, -]$.*

Proof. We could have taken the first statement as a definition in the context of linear algebraic groups and will treat it like that here. The second statement is again most readily seen from the point of view of Lie groups where it is just a variant of the product rule. Take a curve through the identity element i.e. $a(t)$ with $a(0) = Id$ then we have

$$\begin{aligned} \frac{d}{dt}\left(a(t)xa(t)^{-1}\right)_{|t=0} &= a'(0)xa(0) - a(0)^{-1}xa'(0) \\ &= a'(0)x - xa'(0). \end{aligned}$$

□

Remark 9. Now let G be a connected semisimple algebraic group and fix a maximal torus T with Lie algebra $\mathfrak{t}$. Let Ad respectively ad be the adjoint representation of G respectively $\mathfrak{g}$ on $\mathfrak{g}$. According to Remark 8 we can decompose $\mathfrak{g}$ into eigenspaces of T respectively $\mathfrak{t}$.

$$\mathfrak{g} = \mathfrak{g}_0 \bigoplus_{\alpha \in X(T)} \mathfrak{g}_\alpha$$
$$\mathfrak{g} = \mathfrak{g}_0 \bigoplus_{\alpha \in \mathfrak{t}^*} \mathfrak{g}_\alpha$$

Here 0 means the zero element in $X(T)$ respectively in $\mathfrak{t}^*$. By Theorem 2.6.2 we have that this two decompositions conicide with each other, furthermore by the definition of a Cartan subalgebra, see Definition 2.5.2, we have that $\mathfrak{t} = \mathfrak{g}_0$. The $\mathfrak{g}_\alpha$ that are non-zero are called the root-spaces and the corresponding α are called roots. Let R denote the finite set of roots in $\mathfrak{t}^*$ respectively $X(T)$. A character $T \to GL_1$ induces a map $\mathfrak{t} \to \mathfrak{gl}_1 \simeq k$ and can therefore be viewed as an element in $\mathfrak{t}^*$. This allows us to identify $\mathfrak{t}_0^*$, the real span of the set R, and $\mathbb{R} \otimes_{\mathbb{Z}} \mathbb{Z}R$ where $\mathbb{Z}R \subset X(T)$ is the sublattice of $X(T)$ spanned by R. All the following definitions are therefore meant for both cases $R \subset X(T)$ and $R \subset \mathfrak{t}$.

We first summarize the properties of the root space decomposition:

Theorem 2.6.4. *In the situation of Remark 9 we have:*

1) $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$.

2) If $\alpha + \beta \neq 0$ *we have that* $\mathfrak{g}_\alpha$ *and* $\mathfrak{g}_\beta$ *are orthogonal with respect to the Killing form* K.

3) For any $\alpha \in R$ *the Killing form gives a non-degenerate pairing* $\mathfrak{g}_\alpha \otimes \mathfrak{g}_{-\alpha} \to k$.

Proof. See [8, Chapter 6.6]. □

Remark 10. From Theorem 2.6.4 point 3) it follows that the Killing form is non-degenerate on $\mathfrak{t}$ and therefore we can identify $\mathfrak{t}^*$ with $\mathfrak{t}$ via the Killing form. For every $\alpha \in \mathfrak{t}^*$ we thus obtain a $H_\alpha \in \mathfrak{t}$ such that $\alpha = (H_\alpha, -)$. Furthermore the Killing form on $\mathfrak{t}$ induces a non-degenerate bilinear form on $\mathfrak{t}^*$ via $(\alpha, \beta) := (H_\alpha, H_\beta)$. For any $x_\alpha \in \mathfrak{g}_\alpha$ we can choose $y_\alpha \in \mathfrak{g}_{-\alpha}$ with the property $(x_\alpha, y_\alpha) = \frac{2}{\langle \alpha, \alpha \rangle}$. In addition let $h_\alpha := \frac{2H_\alpha}{\langle \alpha, \alpha \rangle}$. One can check that $x_\alpha, y_\alpha, h_\alpha$ generate a subalgebra that satisfies the commutator relation of $\mathfrak{sl}(2, k)$, see Example 2.2.1, and we denote the Lie algebra spanned by this three elements $\mathfrak{sl}(2, k)_\alpha$. We will abuse notation and write $(h, \alpha) := \alpha(h)$ for the natural pairing, this is justified in so far as $\alpha(h) = (h, H_\alpha)$.

Theorem 2.6.5. *Let* $\mathfrak{g}$ *be a semisimple Lie algebra* $\mathfrak{h}$ *a Cartan subalgebra and* R *its root system.*

1) R *spans* $\mathfrak{h}^*$ *as a vector space and the* h_α *(see above) span* $\mathfrak{h}$ *as a vector space.*

2) For each α *the root space* g_α *is one-dimensional.*

3) For any two roots α, β *we have*

$$(h_\alpha, \beta) = \frac{2(\alpha, \beta)}{(\alpha, \alpha)} \in \mathbb{Z}.$$

4) For any $\alpha \in R$ we define $s_\alpha : \mathfrak{h}^* \to \mathfrak{h}^*$ by

$$s_\alpha(\lambda) = \lambda - (h_\alpha, \lambda)\alpha = \lambda - \frac{2(\alpha, \lambda)}{(\alpha, \alpha)}\alpha.$$

Then for any $\beta \in R$ we have that $s_\alpha(\beta)$ is also in R.

5) For any root α the only multiples of α which are also roots are $\pm\alpha$.

6) If α, β are roots such that $\alpha + \beta$ is also a root, then we have $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$.

Proof. See [8, Chapter 6.6]. □

Example 2.6.1. We will continue Example 2.2.1 of $\mathfrak{sl}_2(k)$ here. Note that a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{sl}_2(k)$ is given by the k-span of the element h. Ordering the basis $e_0 = h, e_1 = x, e_2 = y$ we see by the commutator relations that $ad(h)$ takes the following form

$$ad(h) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

and therefore we have

$$\mathfrak{sl}_2(k) = k \cdot h \oplus k \cdot x \oplus ky = \mathfrak{sl}_2(k)_0 \oplus \mathfrak{sl}_2(k)_2 \oplus \mathfrak{sl}_2(k)_{-2}.$$

Here we identified $\mathfrak{h}$ with k via $\lambda \cdot h \mapsto \lambda$ and therefore $\mathfrak{h}^*$ also with k.

This root system is called $A1$ for historical reasons.

We can use this decomposition to completely classify the irreducible representations of $\mathfrak{sl}_2$ and this is actually the main reason to look at rootsystems in general. So let V be an finite dimensional, irreducible representation of $\mathfrak{sl}_2$. We know that the action of the operator h is diagonalizable and we can therefore decompose V into weight spaces

$$V = \bigoplus V_\lambda$$

where λ are the eigenvalues of h. We want to study the action of x, y on the weight spaces. Recall the commutator relations:

$$[h, x] = 2x, [h, y] = 2y.$$

And hence, using the commutator relation $[h, x] = hx - xh$, we have for a vector $v \in V_\lambda$

$$\begin{aligned} h(x(v)) &= x(h(v)) + [h, x](v) \\ &= \lambda x(v) + 2x(v) = (\lambda + 2)x(v) \end{aligned}$$

and similarly we have that $h(y(v)) = (\lambda - 2)y(v)$.This means the eigenvalues of h are congruent to each other modulo 2. We can hence write

$$V = \bigoplus_{n\in\mathbb{Z}} V_{\lambda_0+2n}.$$

Let N be the the final element in the sequence $\lambda_0 + 2n$ for which the weight space is non-empty. Such a number must exist by the finite dimensionality of V. Let $v \in V_N$ we will show next that V is generated by the vectors $v, y(v), y^2(v), y^3(v), \ldots$. First of all an inductive argument using the commutator identity as above shows that

$$x(y^n(v)) = n(N - n + 1)y^{n-1}(v).$$

Now let U be the span of the vectors $v, y(v), y^2(v), y^3(v), \ldots$. We have already seen that U is h invariant and by the above remark it is also x invariant and by definition it is y invariant. Hence U is $\mathfrak{sl}_2$ invariant but V was assumed to be irreducible and therefore $U = V$. Now let M be the smallest number such that $y^M(v) = 0$, again such a number must exist since V is finite dimensional. We get

$$0 = x(y^M(v)) = M(N - M + 1)y^{M-1}(v).$$

And thus $N = M - 1$ which means that N is actually an integer. We make some observations: since the representation is generated by the $y^n(v)$ we see that every V_λ is actually one-dimensional. Since $N = M - 1$ and y shifts a eigenspace by 2 we conclude that the eiegnvalues are symmetric around 0 i.e. we have as eigenvalues:

$$-N, -N + 2, \ldots N - 2, N.$$

We can now classify all the irreducible representations of $\mathfrak{sl}_2$: We first have the trivial representation on $\mathbb{C}$ which is the unique one dimensional irreducible representation. The standard representation of $\mathfrak{sl}_2$ on $V = \mathbb{C}^2$ has eigenvectors e_1 to eigenvalue 1 and e_2 to eigenvalue -1. We can now take $V^{(n)} = Sym^n(V)$ with basis $\{e_1^n, e_1^{n-1}e_2, \ldots, e_1e_2^{n-1}, e_2^n\}$. Although we haven't defined the tensor product of two representations, the action of an element $g \in \mathfrak{g}$ on $x \otimes y$ is given by

$$g(x \otimes y) = g(x) \otimes y + x \otimes g(y).$$

We can thus compute the action of h on $e_1^{n-i}e_2^i$:

$$h(e_1^{n-1}e_2^i) = (n - i)e_1^{n-1}e_2^i - ie_1^{n-1}e_2^i = (n - 2i)e_1^{n-1}e_2^i.$$

And thus we see that $V^{(n)}$ is the unique $n + 1$ dimensional irreducible representation of $\mathfrak{sl}_2$.

Definition 2.6.2. An abstract root system is a finite subset R of a finite dimensional $\mathbb{R}$ or k vector space V together with a non-degenerate bilinear form that satisfies conditions

2), 3), 4) and 5) in Theorem 2.6.5.

Example 2.6.2. In the following we will introduce the root system $A2$ which will serve as an example to visualize the next definitions. This root system is actually the root system associated to $\mathfrak{sl}_3(k)$ and is given by the subset $R \subset \mathbb{R}^2$:

$$R = \left\{ \alpha = \begin{pmatrix} \sqrt{2} \\ 0 \end{pmatrix}, \beta = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{\sqrt{3}}{\sqrt{2}} \end{pmatrix}, \alpha + \beta, -\alpha, -\beta, -(\alpha + \beta) \right\}.$$

We can visualize this system as follows:

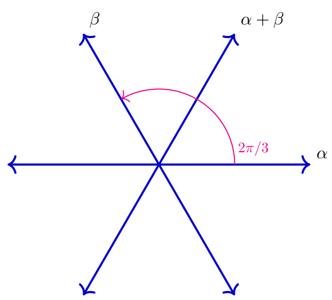


Figure 2.1: The root system A2

One can check that this system indeed forms a abstract root system under the standard scalar product.

Remark 11. Let R be a root system in V. We can choose a set of positive roots R_+ in the following way: for any $\alpha \in R$ only α or $-\alpha$ are contained in R_+ and for any $\alpha, \beta \in R_+$ we have that $\alpha + \beta \in R_+$ if $\alpha + \beta$ is a root. Equivalently we can choose a hyperplane in V that doesn't contain any root and collect all roots on one side of the hyperplane into R_+. The remaining roots are called the negative roots R_-. An element of R_+ is called a simple root, if it cannot be written as the sum of two elements in R_+. The set of simple roots is denoted with Δ and it forms a basis of V. In terms of Example 2.6.2, if we choose the hyperplane to be the antidiagonal we obtain the following picture:

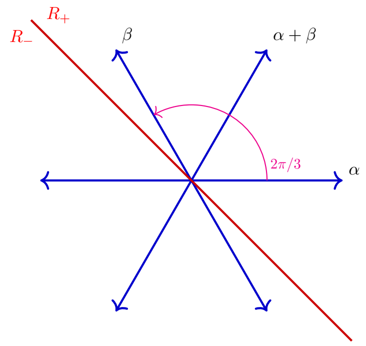


Figure 2.2: The root system A2 with splitting of the roots

Here we have as positive roots $R_+ = \{\alpha, \beta, \alpha + \beta\}$ and as simple roots $\Delta = \{\alpha, \beta\}$.

For any root $\alpha \in R$ we can form

$$\alpha^\vee := \frac{2}{(\alpha, \alpha)}\alpha$$

called the coroot of α. The set of coroots also forms an abstract root system. Now let $\alpha_i^\vee$ be the coroots corresponding to the simple roots, we associate to them a dual basis (ω_i) of V^* i.e. a basis with $\alpha_i^\vee(\omega_j) = \delta_{ij}$. The elements ω_i are called the fundamental weights and their $\mathbb{Z}$ span is called the weight lattice of R^* denoted with P. Elements of the weight lattice are called integral weights. An integral weight is called dominant if it is a non-negative integer combination of the fundamental weights, equivalently we could ask a dominant weight to be of the form $\alpha^i(\omega) \geq 0$ for all simple roots α^i. If we identify V^* with V via the non-degenerate bilinear form we can consider the root lattice Q spanned by all integer combinations of roots as a subset of V^*. In the context of a semisimple Lie algebra $\mathfrak{g}$ this identification sends $\alpha^\vee$ to h_α. By Theorem 2.6.5 point 3) we have $Q \subset P$.

Example 2.6.3. We continue Example 2.6.2. We identify $\mathbb{R}^{2^*}$ with $\mathbb{R}^2$ via the standard scalar product. Taking $\alpha_1 = \alpha$ and $\alpha_2 := \beta$ a quick calculation shows that

$$\omega_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}\sqrt{3}} \end{pmatrix}, \omega_2 = \begin{pmatrix} 0 \\ \frac{\sqrt{2}}{\sqrt{3}} \end{pmatrix}$$

The lattices can be visualized as shown in figure 2.3.

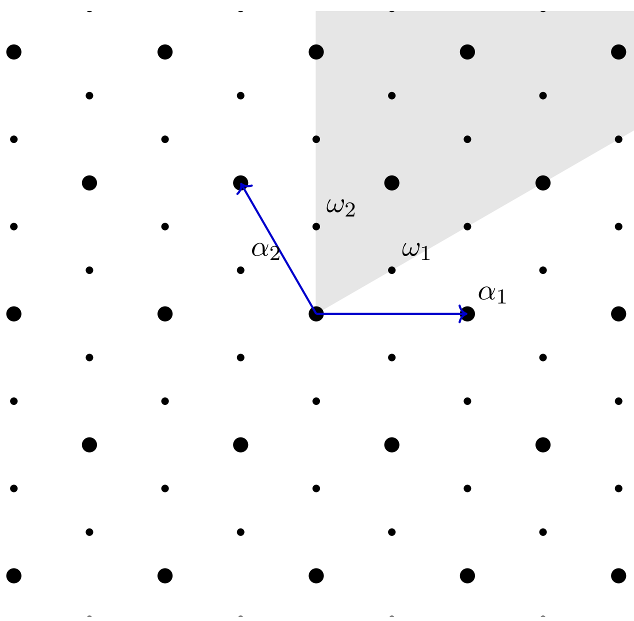


Figure 2.3: The root and weight lattice of A2. Large dots show the root lattice Q small dots show the weight lattice, more precisely $P \setminus Q$. The grey area shows the dominant weights.

Definition 2.6.3. We define a partial ordering on $\mathfrak{h}_0^*$ the real subspace of $\mathfrak{h}$ spanned by the roots in the following way: for $h_1, h_2 \in \mathfrak{h}_0^*$ we say that $h_1 \leq h_2$ if $h_2 - h_1$ is a non-negative linear combination of simple roots $\Delta \subset R_+$. Now let $\mathfrak{g} \to \mathfrak{gl}(V)$ be any finite dimensional representation of a complex semisimple Lie algebra. Recall the definition of the weight space of $\lambda \in \mathfrak{h}^*$:

$$V_\lambda := \{v \in V | \forall x \in \mathfrak{h}, \quad x \cdot v = \lambda(x)v\}.$$

An element $v \in V_\lambda$ is called a weight vector to weight λ and denoted with v_λ. For the finite set $\lambda_1, \ldots, \lambda_m$ of weights with non-empty weight spaces we call λ_i a highest weight if it is maximal with respect to the above partial order.

Example 2.6.4. At this point we have developed enough language and already seen some small examples to have a look on how the theory plays out in a concrete case. We will look at $\mathfrak{sl}_3$ regarded as the 3×3 matrices with trace zero. With the one condition on the diagonal we can see that this is a 8-dimensional space. Our analysis will largely proceed by analogy with the case of $\mathfrak{sl}_2$ i.e. example 2.6.1. The main difference is that now we have a two dimensional space as a cartan subalgebra instead of the space spanned by one operator.

A basis of $\mathfrak{sl}_3$ is given by the elementary matrices e^{ij} for $i \neq j$ which have a one in the ij-th position (friendly reminder: i being the row and j the column) and zeros else and the two matrices

$$h_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, h_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

We will proceed by analogy to the case of $\mathfrak{sl}_2$, i.e. example 2.6.1. We will first analyse the decomposition of the adjoint representation of $\mathfrak{sl}_3$. In the case of $\mathfrak{sl}_2$ we identified a single operator and looked at it's eigenspaces. But now the Cartan subalgebra will no longer be one-dimensional, indeed we will choose it to be

$$\mathfrak{h} = span(h_1, h_2) = \{h \in \mathfrak{sl}_3 | h \text{ is diagonal}\}.$$

We will write an element in $\mathfrak{h}$ as $h = diag(a_1, a_2, a_3)$ to indicate the diagonal elements a_1, a_2, a_3. Recall that the Lie bracket for $\mathfrak{sl}_3$ is given by $[x, y] = xy - yx$ with the usual matrix multiplication. That in mind we can compute the adjoint action of $\mathfrak{h}$ on the elementary matrices e^{ij}:

$$\begin{aligned} [diag(a_1, a_2, a_3), e^{ij}] &= diag(a_1, a_2, a_3)e^{ij} - e^{ij} diag(a_1, a_2, a_3) \\ &= (a_i - a_j)e^{ij}. \end{aligned}$$

By dimension considerations we see that we have a splitting of $\mathfrak{sl}_3$ via the linear functionals $\eta^{ij} := (diag(a_1, a_2, a_3) \mapsto a_i - a_j) \in \mathfrak{h}^*$ in the following way:

$$\mathfrak{sl}_3 = \mathfrak{h} \oplus \bigoplus_{i \neq j} \mathbb{C}e^{ij}.$$

We choose the positive roots to be the η^{ij} associated to the e^{ij} with a one above the diagonal, i.e. $j \leq i$. Concretely we have

$$R_+ = \{\eta^{12}, \eta^{23}, \eta^{13}\} = \{a_1 - a_2, a_2 - a_3, a_1 - a_3\}.$$

In the last equality we abused notation a little to make it easier to spot that $\eta^{12} + \eta^{23} = \eta^{13}$. If we set $\alpha = \eta^{12}, \beta = \eta^{23}$ we recover the rootsystem A2 from example 2.6.2.

To stick to the notation that we developed so far we will denote $\mathbb{C}e^{ij} =: \mathfrak{g}_{ij}$. Now let $x \in \mathfrak{g}_{ij}$ and $y \in \mathfrak{g}_{lk}$ we want to find out what $\mathfrak{h}$ does to $[x, y]$. Let therefore $h \in \mathfrak{h}$, the Jacobi identity yields:

$$\begin{aligned} [h, [x, y]] &= [x, [h, y]] + [[h, x], y] \\ &= (\eta^{ij}(h) + \eta^{lk}(h))[x, y]. \end{aligned}$$

Which is of course the statement of theorem 2.6.4. A quick computation shows that

$[\mathfrak{g}_{12}, \mathfrak{g}_{23}] = \mathfrak{g}_{13}$ and indeed this holds whenever the sum of two roots is again a root and two rootspaces annihilate each other whenever the sum of their roots is not a root again. The exception of this rule is for when we pair a rootspace with it's negative, then we always have $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subset \mathfrak{h}$ which of course is again consistent with theorem 2.6.4 point 3). The other properties of a rootsystem can now be checked by direct computation but will be omitted here.

We now turn our attention to finite dimensional irreducible representations of $\mathfrak{sl}_3$. Let V be such a representation. Recall that as in remark 8 we can decompose V into weight spaces

$$V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda.$$

Now let $\alpha \in R$ be a root, we want to see how the rootspace $\mathfrak{g}_\alpha$ acts on $v \in V_\lambda$. Let $h \in \mathfrak{h}$ and $x \in \mathfrak{g}_\alpha$ we get:

$$\begin{aligned} h(x(v)) &= x(h(v)) + [h, x](v) \\ &= (\lambda(h) + \alpha(h))x(v) \end{aligned}$$

and thus $x(v)$ is weight vector of weight $\lambda + \alpha$, i.e. g_α acts in the form $V_\lambda \to V_{\lambda+\alpha}$. Since the representation is irreducible and $\mathfrak{h}$ fixes the weight spaces we must reach every weight space from another weight space by the action of some rootspace, see below for more details. We thus have shown that the difference of any two weights is a linear combination of some roots. Or in a more technical way we could say that the weights are congruent to each other modulo the root lattice. In particular we see that the rootspaces associated to the positive roots send a weight to a weight which is greater with respect to the order introduced above and since their are only finitely many weights there has to be a highest weight λ of our representation. The associated weight space V_λ gets annihilated by all the positive rootspaces.

We will now show that V is generated by a highest weight vector $v \in V_\lambda$ under the action of the negative root spaces. This can be seen as follows. Let U be the span of all the vectors which are reached by repeatedly applying e^{21}, e^{31}, e^{32}. Now let U_i be the span of the vectors which are generated by at most i applications of these operators, so for example $e^{21}(e^{32}(v)) \in U_2$. Clearly we have that $U = \cup U_i$. By the above considerations we have that $\mathfrak{h}$ fixes every U_i since the translates of weight vectors are weight vectors again. Furthermore the positive rootspaces send U_i to U_{i-1}. We illustrate this by the example $e^{21}(e^{32}(v)) \in U_2$, the general argument then uses induction. Applying e^{12} to

$e^{21}(e^{32}(v))$ we get, using the commutator identity:

$$\begin{aligned} e^{12}(e^{21}(e^{32}(v))) &= e^{21}(e^{12}(e^{32}(v))) + \underbrace{\underbrace{[e^{21},e^{12}]}_{\in\mathfrak{h}}(e^{32}(v))}_{\in U_1} \\ &= e^{21}(e^{32}(\underbrace{e^{21}(v)}_{=0})) + e^{21}(\underbrace{[e^{32},e^{21}]}_{=0}(v)) + U_1 \end{aligned}$$

and therefore $e^{12}(e^{21}(e^{32}(v))) \in U_1$. It follows that U is invariant under the action of the positive rootspace as well as the action of $\mathfrak{h}$ and by definition as well under the action of the negative rootspaces, thus U is $\mathfrak{sl}_3$ invariant and by the irreducibility of V we have that $U = V$.

Two observations are immediate now: firstly the weightspace of the highest weight vector v is one-dimensional since all the translates of the action of the negative rootspaces have lower weight. And secondly, by the same reasoning, the highest weight vector is unique up to scalar multiplication. Recall that the order we defined is only partial and hence, a priori, it could be that there are two vectors of highest weight of an irreducible representation. But again, all the translates of one highest weight vector are of lower weight and generate V and hence this is impossible.

We can now check that the highest weight is indeed dominant. We will use our knowledge of $\mathfrak{sl}_2$ representations to achieve this. Recall that in any semisimple Lie algebra we have subalgebras isomorphic to $\mathfrak{sl}_2$ as discussed in remark 10. In our concrete case we have for example that the triple $e^{13}, e^{31}, h_1 + h_2$ span a subalgebra isomorphic to $\mathfrak{sl}_2$. Indeed if we set $h = h_1 + h_2, x = e^{13}, y = e^{31}$ a direct computation shows that we have the commutator relations

$$[x,y] = h, [h,x] = 2x, [h,y] = 2y.$$

To shorten notation in the following we will denote with α a root and then $\mathfrak{sl}_\alpha \simeq \mathfrak{sl}_2$ will denote the subalgebra spanned by $\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}, [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subset \mathfrak{h}$.

We will now convince ourself first that the highest weight λ is indeed a weight i.e. it is contained in the weight lattice. By remark 11 this means that we need to check that if $\alpha \in R_+$ is a simple root then $\lambda(h_\alpha) \in \mathbb{Z}$. But $\mathfrak{sl}_\alpha$ acts on V_λ by restriction. If we denote with $U \subset V$ the irreducible representation of $\mathfrak{sl}_\alpha$ which is generated by the action of $\mathfrak{sl}_\alpha$ on V_λ and then consider our knowledge about irreducible representations of $\mathfrak{sl}_2$ it is immediate that $\lambda(h_\alpha)$ is an integer. Furthermore since λ also gets annihilated by $\mathfrak{g}_\alpha$ and therefore U gets generated by powers of $g_{-\alpha}$. Recalling that the weights of a $\mathfrak{sl}_2$ representation are symmetric around zero and that $\mathfrak{g}_{-\alpha}$ sends a weight vector to a vector of strictly smaller weight we can conclude that $\lambda(h_\alpha) \geq 0$. Since this holds for all α we have proven that λ is indeed a dominant weight.

In principle we could now give a complete classification of the irreducible representations of $\mathfrak{sl}_3$, but we will only outline the rough idea here: Let ω_1, ω_2 be the fundamental weights as in example 2.6.2. Now the natural representation on $V = \mathbb{C}^3$ of $\mathfrak{sl}_3$ has weight ω_2 and its dual V^* has weight ω_1. From this we can generate an irreducible representation of weight $l\omega_1 + k\omega_2$ for $l, k \in \mathbb{N}$ simply by taking $Sym(V)^{\otimes k} \otimes Sym(V^*)^{\otimes l}$.

Most of the above observations didn't actually use the structure of $\mathfrak{sl}_3$ but only the semisimplicity of $\mathfrak{sl}_3$. That all of these phenomena generalize to arbitrary semisimple Lie algebras is summarized in the following theorem.

Theorem 2.6.6 (Theorem of highest weight)**.** *Let $G \to GL(V)$ be a finite dimensional irreducible representation of a connected semisimple algebraic group G respectively of a semisimple Lie algebra $\mathfrak{g}$. Then the following holds:*

1) *All weights of V lie in $\mathfrak{t}_0^*$.*
2) *V has a highest weight.*
3) *The highest weight is always a dominant weight.*
4) *Two irreducible representations with the same highest weight are isomorphic.*
5) *Every dominant weight is the highest weight of a irreducible representation.*

Proof. See for example [9, Chapter 14.1] or [4, P. 468 Theorem 30.5.2 and Theorem 30.5.3] □

In addition we know how the rootspaces act on the weightspaces for any representation V of $\mathfrak{g}$.

Theorem 2.6.7. *Let V be a representation of $\mathfrak{g}$. Fix $\mathfrak{t}$ as above and let V_λ be a weight space of V. Let $\mathfrak{g}_\alpha$ be a rootspace with respect to $\mathfrak{t}$ then it holds that*

$$\mathfrak{g}_\alpha V_\lambda \subset V_{\lambda+\alpha}.$$

In particular if $\alpha \in R_-$ we have that $\mathfrak{g}_\alpha$ sends V_λ into a weightspace of lower weight.

Proof. Let $x_\alpha \in \mathfrak{g}_\alpha$ and $v_\lambda \in V_\lambda$. Let $t \in \mathfrak{t}$, an application of the commutator identity $[x, y] = xy - yx$ yields:

$$\begin{aligned} t \cdot x_\alpha \cdot v_\lambda &= [t, x_\alpha]v_\lambda + x_\alpha \cdot t \cdot v_\lambda \\ &= \alpha(t)x_\alpha v_\lambda + \lambda(t)x_\alpha v_\lambda. \end{aligned}$$

□

Lastly we will give an important and famous lemma which we will need later on.

Theorem 2.6.8 (Schur's lemma)**.** *Let G be an algebraic group and V, W be irreducible representations of G. Let $f : V \to W$ be a morphism of representations, i.e. a linear* *map from V to W such that $\sigma_W(g) \circ f = f \circ \sigma_V$. Then either $V = W$ and $f = \lambda \cdot Id$ for a scalar $\lambda \in k$ or V and W are not isomorphic and f is the zero map.*

Proof. Suppose f is non-zero and let $V' \subset V$ be the kernel of f. For $v \in V'$ we have that $f(\sigma_V(g)(v)) = \sigma_W(g)(f(v)) = 0$ and hence V' is a subrepresentation of V, by the irreducibility of V and since f is nonzero that means that $V' = 0$ and hence f is injective. The surjectivity is similar: By the G equivariance we have that $f(V)$ is G invariant and hence a subrepresentation of W. By assumption f is nonzero and hence $f(V) = W$ and all in all we have that $V = W$.
Now let λ be a eigenvalue of f. The map $f' = f - \lambda Id$ is again equivariant but it has a nonzero kernel, namely any eigenvector to eigenvalue λ. But then by the above discussion we have $f' = f - \lambda Id = 0$ and hence $f = \lambda Id$. □

2.7 More about Borel subgroups, parabolic subgroups and Unipotent groups

Example 2.7.1. Let $\mathfrak{g}$ be the semisimple Lie algebra of a connected semisimple algebraic group and fix a maximal torus T. Let $\mathfrak{t} = Lie(T)$. Let R be the associated root system and R_+ resp. R_- be the positive resp. negative roots. Then we have that

$$\begin{aligned} \mathfrak{b}_+ &= \mathfrak{t} \oplus \bigoplus_{\alpha \in R_+} \mathfrak{g}_\alpha \\ \mathfrak{b}_- &= \mathfrak{t} \oplus \bigoplus_{\alpha \in R_-} \mathfrak{g}_\alpha \end{aligned}$$

are subalgebras by Theorem 2.6.4 point 1). Again by Theorem 2.6.4 point 1) and the definition of positive resp. negative roots we have that these are solvable and by Remark 10 about $\mathfrak{sl}_2(k)$ triple we can see that these are indeed maximal with respect to being solvable. Therefore $\mathfrak{b}_+, \mathfrak{b}_-$ are Borel subalgebras. The associated Borel subgroups, see Theorem 2.5.2, are denoted with B_+ or sometimes just B and B_-.

To be somewhat more concrete we consider the case of SL_n As a maximal torus we choose the diagonal matrices $T = \{(a_{ij})_{ij} | a_{ii} \neq 0 \text{ and } a_{ij} = 0 \text{ for } i \neq j\} \subset SL_n$. It's lie algebra $Lie(T) = \mathfrak{t}$ are the diagonal matrices inside the $n \times n$ matrices with zero trace. Let e^{ij} be the matrix with an 1 in the ij-entry and zeros elsewhere. We denote with $diag(a_1, \ldots, a_n)$ the diagonal matrix with a_i on the diagonal. A quick computation shows that

$$[diag(a_1, \ldots, a_n), e^{ij}] = (a_i - a_j)e^{ij}.$$

And hence the matrices e^{ij} form a basis of eigenvectors together with the diagonal matrices themself. We can let R be the linear functionals $diag(a_1, \ldots, a_n) \mapsto a_i - a_j$ and R_+ the functionals with $i \geq j$. This shows that in this case $\mathfrak{b}$ is equal to the space of upper triangular matrices with trace zero and indeed that $B \subset SL_n$ is equal to the upper triangular matrices with determinant 1.

Theorem 2.7.1. *Let G be a connected semisimple algebraic group and fix a torus T. Let R be the associated root system. Define*

$$\begin{aligned} \mathfrak{n}_+ &= \bigoplus_{\alpha \in R_+} \mathfrak{g}_\alpha \\ \mathfrak{n}_- &= \bigoplus_{\alpha \in R_-} \mathfrak{g}_\alpha . \end{aligned}$$

Then $\mathfrak{n}_+, \mathfrak{n}_-$ are nilpotent subalgebras of $\mathfrak{g}$ and there exist connected unipotent subgroups U_+ and U_- of G with $Lie(U_+) = \mathfrak{n}_+$ and $Lie(U_-) = \mathfrak{n}_-$ and furthermore

$$B_+ = TU_+ \qquad B_- = TU_- .$$

Proof. The assertion about $\mathfrak{n}_+, \mathfrak{n}_-$ being nilpotent subalgebras follows from Theorem 2.6.4 point 1). For the rest see [4, P. 409, Chapter 26.5]. □

Example 2.7.2. The main example we have in mind here is essentially Example 2.7.1. The group being SL_n and the torus T being the diagonal matrices in SL_n. The upper triangular matrices are B_+ and the lower triangular matrices being B_- then U_+ is the upper triangular matrices with ones on the diagonal and respectively U_- the lower triangular matrices with ones on the diagonal.

Now let $\mathfrak{u} \subset \mathfrak{sl}_n$ be the strict upper triangular matrices and $U \subset SL_n$ the upper triangular matrices with ones on the diagonal. We define a map from $\mathfrak{u} \to U$ by setting

$$x \mapsto \exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

Here we consider $[u)$ as matrices and x^n just means the usual matrix product. By the nilpotency of x we see that this is indeed just a finite sum and hence an algebraic map. That $\exp(x)$ is indeed unipotent is easy to see, since $\exp(x) = id + x + x^2/2 \ldots$ and the product of strict upper triangular matrices is strict upper triangular. In the case of SL_2 this map is particularly easy to understand since for $x \in \mathfrak{u}$ we have that

$$x = \begin{pmatrix} 0 & \lambda \\ 0 & 0 \end{pmatrix} \mapsto \exp(x) = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$$

and we can see that $\exp : \mathfrak{u} \to U$ is indeed an isomorphism. This holds more generally and is summarized in the following theorem.

Theorem 2.7.2. *Let $U \subset GL(V)$ be a connected unipotent group and $\mathfrak{u} = Lie(U) \subset \mathfrak{gl}(V)$ its Lie algebra. If we view $\mathfrak{gl}(V)$ as linear maps from $V \to V$ then it holds that all elements $x \in \mathfrak{u}$ are nilpotent in the usual sense i.e. there exists a $n \in \mathbb{N}$ such that $x^n = 0$ and furthermore $\mathfrak{u}$ is isomorphic to U as a variety via the exponential map*

$$\begin{array}{rcl} \mathfrak{u} & \to & U \\ x & \mapsto & \exp(x) = \sum_{k=0}^{n} \frac{x^k}{k!}. \end{array}$$

Proof. See for example [10]. □

Example 2.7.3. We turn to $U \subset SL_n$ the upper triangular matrices again. Let V be a representation of U and consider $\mathbb{P}(V) = V \subset 0/ \simeq$ the projectivization of V. The action of U on V descents to an action on $\mathbb{P}(V)$ since it is linear and U consists of invertible matrices. Let $x \in \mathbb{P}(V)$ be any point, we want to understand how the orbit $U \cdot x$ looks like. Let $v \in V$ be a representant of x, i.e. $\overline{v} = x$. We consider the special case of $n = 3$ for illustration purposes here: Let $v = (v_1, v_2, v_3)$ then

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} + \begin{pmatrix} av_2 + bv_3 \\ cv_3 \\ 0 \end{pmatrix}$$

and hence $U \cdot v$ is an affine subspace of V and in particular it is closed and we can see that $U \cdot x$ is also closed. Again this holds more generally and is summarized in the next theorem.

Theorem 2.7.3 (Rosenlicht). *Let U be a unipotent linear algebraic group acting on affine variety X. Then all orbits of U in X are closed*

Proof. See [5, P. 88, Proposition 4.10] □

Example 2.7.4. We consider the case of SL_3 with Lie algebra $\mathfrak{sl}_3$ and it's natural representation $V = \mathbb{C}^3$ next. The highest weight vector being the vector $e_1 = (1, 0, 0)$, see example 2.6.4. The subgroup of SL_3 that fixes the line $\mathbb{C}e_1$ consist of the matrices

$$P = \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \right\} \subset SL_3.$$

By remark 4 the quotient SL_3/P is equal to the orbit $G \cdot \overline{e_1} \subset \mathbb{P}(V) \simeq \mathbb{P}^2$. But the action of G on $\mathbb{P}^2$ is transitive since it is transitive on $V \setminus 0$ and hence $G/P \simeq \mathbb{P}^2$ which means that P is actually a parabolic subgroup. We note that the Lie algebra of P is precisely

$$Lie(P) = \mathfrak{t} \bigoplus_{\alpha \in R_+} \mathfrak{g}_\alpha \oplus \mathfrak{g}_{32}.$$

The last summand being the rootspace to the root η^{32}, see example 2.6.4 for a reminder. In the language of example 2.6.2 this this is the simple root $-\alpha_1$. The weight of e_1 is ω_2 and we can see that $(\alpha_1, \omega_2) = 0$. That this is no coincidence is part of the next theorem.

Theorem 2.7.4. *Let G be connected semisimple with maximal torus T and Borel subgroup B corresponding to the positive roots*

$$\mathfrak{b} = \mathfrak{t} \bigoplus_{\alpha \in R_+} \mathfrak{g}_\alpha.$$

Now let V be a highest weight representation of dominant weight λ with highest weight vector v_λ. Then the subgroup P_λ fixing the line through v_λ is parabolic. P_λ contains B and its Lie algebra is generated by $\mathfrak{b}$ and the $\mathfrak{g}_{-\alpha}$ with α is simple and $(\alpha, \lambda) = 0$. Call the subset of such simple roots S, we therefore have

$$Lie(P_\lambda) = \mathfrak{p} = \mathfrak{t} \bigoplus_{\alpha \in R_+} \mathfrak{g}_\alpha \bigoplus_{\alpha \in S} \mathfrak{g}_{-\alpha}.$$

Furthermore the stabilizer group of a line $L = k \cdot v$ is parabolic if and only if v is a highest weight vector.

Proof. See [9, P. 382–395] □

2.8 The Weyl group

Remark 12. Fix G a connected semisimple group and T a maximal torus in G. Let $N_G(T)$ be the normalizer of T in G. We have an action of $N_G(T)$ on the character lattice $X(T)$ in the following way: let $n \in N_G(T)$ and $\chi \in X(T)$ then since $n^{-1}tn \in T$ for all $t \in T$ we have that

$$\chi(n^{-1}tn)$$

is again a character of T. Notice that this action factors through T i.e. if $n, n' \in N_G(T)$ with $n = t'n'$ for some $t' \in T$ then

$$\chi(n^{-1}tn) = \chi(n'^{-1}t'^{-1}tt'n') = \chi(n'^{-1}tn')$$

by the commutativity of T. We thus obtain a action of $W := N_G(T)/T$ on $X(T)$ and W is called the *Weyl group* of T. It can be shown that the Weyl group is always finite, in fact it coincides with the group generated by reflections as in Theorem 2.6.5 point 4). See [15] for a full discussion. From this remark it is clear that W sends roots to roots but it can also be seen from theorem 2.8.1 below. For an element $w \in W$ and a character $\chi \in X(T)$ we denote with $w(\chi)$ the character induced by w.

Example 2.8.1. Let $G = SL_n$ and T be the diagonal matrices inside SL_n. A calculation shows that $N_G(T)$ coincides with the generalized permutation matrices i.e. matrices with one entry in every row and column such that the product of the entries is equal to one. Indeed let p be such a generalized permutation matrix. Then p^{-1} is again a generalized permutation matrix. And a straightforward computation shows that for any diagonal matrix $d \in T$ we have that ptp^{-1} is again diagonal.

For the other direction let $x \in N_G(T)$. Let $V = \mathbb{C}^n$ be the natural representation of SL_3. A set of common eigenvectors to T is given by the canonical basis $\{e_1, \dots, e_n\}$. For any $t \in T$ we have

$$t(x(e_i)) = x(t(e_i)) = x(\lambda e_i) = \lambda x(e_i),$$

which means that $x(e_i)$ is again an eigenvector. This means $x(e_i)$ is a multiple of one of the e_j and hence x can only have one entry per column. Furthermore since x is invertible the non-zero entries have to be in different rows too which can be seen by computing the determinant of x.

After determine $N_G(T)$ we can see that $W = N_G(T)/T$ can be identified with S_n.Since we have the condition on the determinant being 1 we have to distinguish two cases: for an even permutation w we can pick a representative in $N_G(T)$ to be the permutation matrix with ones in every non-zero entry and for an odd permutation we can pick a representative that has to have a -1 in exactly one of the non-zero entries.

Theorem 2.8.1. *Let V be a representation of G. Fix a maximal torus T and consider its weight decomposition*

$$V = \bigoplus V_\chi.$$

Let $w \in W$ and pick a representative $n \in N_G(T)$ such that $\overline{n} = w$. Then

$$n \cdot V_\chi = V_w(\chi).$$

Proof. Let $t \in T$ and $v \in V_\chi$. We have

$$t \cdot (n \cdot v) = n \cdot (n^{-1}tn \cdot v) = n \cdot \chi(n^{-1}tn)v = w(\chi)(t) \cdot n \cdot v.$$

□

Example 2.8.2. We will look into an important application of the Weyl group here. First let $G = SL_2$ and as above let B be the upper triangular matrices in SL_2 as well as T being the diagonal matrices. Let $W = N_G(T)/T$ be the Weyl group associated to T. By example 2.8.1 the group W can be represented by the matrices

$$w_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, w_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

We will have a look at the double cosets BwB for $w \in W$. First of all we should make clear that we willingly confuse notation here. Strictly speaking we should take a representative of w in $N_G(T)$ but since for any $t \in T$ we have that $tB = B$ we have that if $w = w't$ then $wB = w'tB = w'B$ and hence the double cosets are independent of the representative.

Now for w_0 we obviously have that $Bw_0B = BB = B$. Lets have a look at the double coset Bw_1B for a $b_2 \in B$ we have

$$w_0 b_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} 0 & -c \\ a & b \end{pmatrix}.$$

And thus we have

$$b_1 w_1 b_2 = \begin{pmatrix} d & e \\ 0 & f \end{pmatrix} \begin{pmatrix} 0 & -c \\ a & b \end{pmatrix} = \begin{pmatrix} ae & -cd + be \\ af & bf \end{pmatrix}.$$

First of all we note that $Bw_0B = B$ is disjoint from Bw_1B. Indeed if af is a zero entry then either a or f is zero but then either b_1 or b_2 would have determinant zero which is impossible. A direct computation shows that any other matrix in $SL_2 \setminus B$ can be reached as a product $b_1w_1b_2$ and thus we have a disjoint union

$$G = Bw_0B \cup Bw_1B.$$

This is the so called Bruhat decomposition of G. For the group GL_n this is a representation of the row reduction algorithm known from linear algebra. The rightmost B contains the echelon form of a given matrix, the w accounts for permutation of rows that have to be made to get to the echelon form and the leftmost B contains the coefficient matrix of the Gauss-Jordan elimination algorithm. So in very concrete sense the Bruhat decomposition can be regarded as a generalization the Gauss-Jordan decomposition of a matrix.

We will look into a refinement of this decomposition. Note that, for example, in the case of GL_n the decompositon of a given matrix a into $a = b_1wb_2$ is far from unique. One could multiply b_1 by a scalar λ and b_2 by $1/\lambda$ and get another representation. For simplicity we turn to the case of SL_2 again. Let U resp. U^- denote the upper resp. lower triangular matrices with one's on the diagonal. Define

$$U_w = U \cap wU^-w^{-1}.$$

A straightforward calculation shows that in this case $U_{w_0} = e$ and $U_{w_1} = U$. Define a map

$$\begin{aligned} \alpha_w : U_w \times B &\to BwB \\ (x,y) &\mapsto xwy. \end{aligned}$$

Now its easy to see that α_{w_0} is actually an isomorphism, just sending $b \in B$ to itself. Lets have a look at α_{w_1}. Let $u \in U$ and $b_2 \in B$, we get

$$uw_1b_2 = \begin{pmatrix} 1 & e \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -c \\ a & b \end{pmatrix} = \begin{pmatrix} ae & -c+be \\ a & b \end{pmatrix}.$$

And a direct comparison with the matrix

$$\begin{pmatrix} ae & -cd+be \\ af & bf \end{pmatrix}$$

shows that this is again an isomorphism. This is of course not that surprising in this case since we could have taken the matrix b_1 from above and multiply it with tt^{-1} where

$$t = \begin{pmatrix} d^{-1} & 0 \\ 0 & f^{-1} \end{pmatrix}.$$

and since $t^{-1}w = wt'$ for some $t' \in T$ we have that $b_1twt'b_2 = b_1wb_2$ but b_1w is actually an element in U. This is „normalizing“ the choice of the lefthandside in the Bruhat decomposition.

Theorem 2.8.2 (Bruhat decomposition)**.** *Let G be connected and semisimple and fix a maximal torus T. Let U^-, U and B be as in Theorem 2.7.1. Denote with $W = N_G(T)/T$ the Weyl group associated to T. Then it holds that*

$$G = \cup_{w \in W} BwB.$$

Furthermore define $U_w := U \cap wU^-w^{-1}$. We have an isomorphism of varieties given by

$$\begin{aligned} U_w \times B &\to BwB \\ (x,y) &\mapsto xwy. \end{aligned}$$

Furthermore there is an open covering of G given by

$$G = \cup_{w \in W} wU^-TU.$$

By the irreducibility of G we have that wU^-TU is dense.

Proof. See [5, Chapter 14.12] or [11, Theorem 22.72] for a full proof of the first part. For the second part observe that

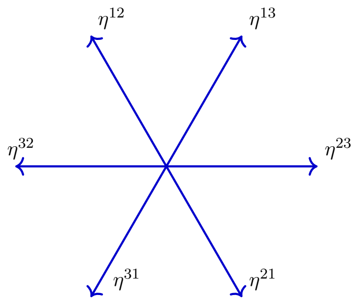
$$wU^{-}TU = wU^{-}w^{-1}wTU \supset U_w wTU = U_w wB = BwB.$$

Here the containment follows from the definition of U_w and the last equality is a consequence of the first part of the theorem. For the openness of $wU^{-}TU$ see [11, Theorem 22.72]. □

Example 2.8.3. We want to understand next how the Weyl group acts on roots. As already mentioned in remark 12 the Weyl group is isomorphic to the group that is generated by the reflections as in theorem 2.6.5. But we want to see this in a more direct way. Let $G = SL_3$, we consider the adjoint action of SL_3 on $\mathfrak{sl}_3$. Recall that this action is given by axa^{-1} for $a \in SL_3$ and $x \in \mathfrak{sl}_3$. Now let $\sigma \in W = N_G(T)/T$, here we will interpret σ as a permutation matrix. There is a minor issue with odd permutations, since there has to be a -1 in one of the entries but we will ignore this for now. Since a product $\sigma \cdot x$ will only permute the rows of a matrix and a product $x\sigma^{-1}$ permutes the columns we have for an elementary matrix e^{ij}

$$\sigma e^{ij} \sigma^{-1} = e^{\sigma(i)\sigma^{-1}(j)}.$$

Here we identified the permutation matrix σ with the corresponding permutation $\sigma \in S_3$. Thus we get that the induced action of W on the roots sends η^{ij} to $\eta^{\sigma(i)\sigma^{-1}(j)}$. We focus on the the elements $(12), (23) \in S_3$ next since those elements generate the whole of S_3. To visualize what is happening we take the description of the rootsystem of $\mathfrak{sl}_3$.



From this picture we can directly read of what happens when (12) is applied to the root system: This corresponds to a reflection along the orthogonal complement of the root η^{12} and similarly (23) acts as a reflection along the orthogonal complement of $\eta^{2,3}$. Of course the same is true for $(1,3)$ and since any 3-cycle is a product of two 2-cycles and a product of two reflections is a rotation we now fully understand the action of the Weyl group on $A3$. What is important to us is the following fact: If we take any basis B of the root lattice, then $w(B) \neq B$. Note that it is important that B is a basis of the lattice, η^{12}, η^{13} is a basis for the vector space but not for the root lattice and indeed the permutation (23) fixes this set of vectors.

Theorem 2.8.3. *Let $\Delta \subset R$ be a basis of the root lattice. If $w(\Delta) = \Delta$ then $w = id$.*

Proof. See for example [4, P. 248, Theorem 18.8.7]. □

2.9 About the classification of semisimple algebraic groups

Remark 13. We will give a brief overview over the classification of semisimple algebraic groups. Let G be a connected semisimple algebraic group and fix a maximal torus T. Denote with $R \subset X(T)$ the corresponding rootsystem. Let $Q = \mathbb{Z}R \subset X(T)$ be the rootlattice and consider $E = \mathbb{R} \otimes_{\mathbb{Z}} Q$. Now let P be the weight lattice in E, see Remark 11 for a reminder. We can embed $X(T)$ into E and the classification of semisimple algebraic groups states that G is uniquely determined by the relation $Q \subset X(T) \subset P$. In addition fixing a rootsystem with corresponding rootlattice Q and weightlattice P for every sublattice R with $Q \subset R \subset P$ there exists a connected semisimple algebraic group with $R = X(T)$ and Q, P being its root respectively its weight lattice.

Definition 2.9.1. Let G be a semisimple connected group, and suppose that $Q \subset X(T) \subset P$ are as in remark 13. If $X(T) = P$, then G is called *simply connected* and if $X(T) = P$ then G is called *adjoint.*

Theorem 2.9.1. *Fix a root lattice Q and weight lattice P let G_{sc} be the simply connected group corresponding to this root system and G_{ad} be the adjoint group corresponding to it. Let G be any other connected semisimple group of the same type. Then we have an inclusion $X(T) \to P = X(T_{sc})$ and this induces a morphism*

$$G_{sc} \twoheadrightarrow G$$

with kernel isomorphic to $P/X(T)$ and similarly

$$G \twoheadrightarrow G_{ad}$$

with kernel $X(T)/Q$. In particular the kernel is finite because the lattices have the same rank. Furthermore the morphism

$$G_{sc} \twoheadrightarrow G_{ad}$$

has kernel P/Q and P/Q is isomorphic to the center of G_{sc}, thus G_{ad} can be realized as $G_{sc}/Z(G_{sc})$.

Proof. See [12]. □

Example 2.9.1. We have seen the root and weight lattice of $\mathfrak{sl}_3$ in example 2.6.2 and following. The quotient P/Q is isomorphic to $\mathbb{Z}/3\mathbb{Z}$ as the picture of the respective lattices shows. Since $\mathbb{Z}/3\mathbb{Z}$ has no non-trivial subgroups there are only two groups of this type. Indeed we have a morphism

$$SL_3 \to PSL_3 \simeq SL_3/Z(SL_3)$$

showing that SL_3 is simply connected and PSL_3 is of adjoint type. A closer look at the center of SL_3 shows that these consist of the diagonal matrices $\lambda \cdot Id$ where λ being a third root of unity, confirming that $Z(SL_3) \simeq \mathbb{Z}/3\mathbb{Z}$. This holds more generally: one can compute that the the quotient P/Q is isomorphic to $\mathbb{Z}/n\mathbb{Z}$ for $\mathfrak{sl}_n$ which is precisely the center of SL_n. So we have that SL_n is simply connected and PSL_n is the associated group of adjoint type. For a full classification of semisimple linear algebraic groups see [6].

3 The compactification method

Throughout this whole chapter we follow the exposition of C. de Concini and C. Procesi [1] and the lecture notes of Sam Evens and Benjamin F. Jones [2]. In particular the notation and the line of reasoning follows the latter.

3.1 A special representation

Remark 14. In the following we will work with simply connected algebraic groups. The reason is basically the theorem of the highest weight 2.6.6: Note that all the algebraic groups with the same root system have the same Lie algebra but the representation theory only coincides for the groups of simply connected type. This is because when $X(T)$ is strictly contained in the weight lattice Q there are irreducible representations of $\mathfrak{g}$ that do not correspond to representations of G. Just look at the natural representation of SL_3 on $V = \mathbb{C}^3$ for an example of this. Since the center of SL_3 acts non trivially on V this representation doesn't correspond to a representation of PSL_3.

Although our construction will use the group of simply connected type our final construction will hold only for the group of adjoint type. The reason is explained in the following theorem which is crucial to the construction.

Definition 3.1.1. Let ω_i be the fundamental weights, see Remark 11. A dominant weight $\lambda \in Q$ is called regular if it is of the form

$$\lambda = \sum n_i \omega_i$$

with $n_i \in \mathbb{N}_{>0}$.

Theorem 3.1.1. *Let G be a simply connected algebraic group with root datum R relativ to a maximal torus T. Then the highest weight representation $V(\lambda)$ with regular dominant weight λ satisfies the following properties*

1) *The weigth space V_λ is one dimensional.*
2) *$g_{-\alpha} V_\lambda = V_{\lambda-\alpha}$ is one dimensional for all positive roots α and in general all weights of $V(\lambda)$ are of the form $\lambda - \sum n_i \alpha_i$ for the simple roots α_i.*
3) *The stabilizer of V_λ is B^+.*
4) *The stabilizer of $V_\lambda^* \subset V(\lambda)^*$ is B^-.*
5) *The action of G on $\mathbb{P}(V(\lambda))$ factors through a faithful action of G_{ad}.*

Proof. The existence of $V(\lambda)$ is Theorem 2.6.6. Assertion 1) and 2) are consequences of the Verma module construction and the viewpoint of irreducible representations as quotients of Verma modules. The construction can be found in [4, P. 465, Chapter 30.4].

Point 3) uses the fact that λ is regular and Theorem 2.7.4: The parabolic subgroup P fixing V_λ has the property, that $Lie(P)$ is spanned by $\mathfrak{b}$ and the $\mathfrak{g}_{-\alpha}$ for simple α with $(\lambda, \alpha) = 0$. Note that the last equation is equivalent to $(\lambda, \alpha^\vee) = 0$ but $\lambda = \sum n_i \omega_i$ and therefore if α_i is simple we have $(\lambda, \alpha_i^\vee) = n_i$. We conclude that $Lie(P) = \mathfrak{b}$ and therefore $P = B$.

Point 4) follows from the observation that V_λ^* has weight $-\lambda$ in $V(\lambda)^*$ and this is in fact a lowest weight in $V(\lambda)^*$. Now just reversing the choice of positive roots makes $-\lambda$ a highest weight of $V(\lambda)^*$. Now just apply the same reasoning as in 3) to B^- to obtain 4).

For 5) let $z \in Z(G)$, then $z : V(\lambda) \to V(\lambda)$ is a morphism of representations i.e. G-equivariant and therefore by Schur's lemma it acts as a scalar. Another way to see this, which uses the language we developed so far and is useful later is as follows: Since G is semisimple we have that $z \in T$. We make some quick observations about the characters. By the above discussion we have that Q is equal to the characters of $G/Z(G) = G_{ad}$ and thus a character in Q has to be trivial on $Z(G)$. Let μ be any weight of $V(\lambda)$, by 2) $\mu - \lambda$ is of the form $\mu - \lambda = \sum n_i \alpha_i$ where α_i are the simple roots and thus $\mu - \lambda \in Q$. Now fix a base $v_0, \ldots, v_n$ of weight vectors of $V(\lambda)$ with v_0 of weight λ and v_i of weight μ_i. We get

$$\begin{aligned} z \cdot v &= z \cdot \left(\sum b_i v_i\right) \\ &= b_0 z^\lambda + \sum b_i z^{\mu_i} v_i \end{aligned}$$

But now multiplying with $1 = z^\lambda z^{-\lambda}$ and using that $z^{\mu_i - \lambda} = 1$ we obtain

$$\begin{aligned} z \cdot v &= z^\lambda \left(b_0 v_0 + \sum b_i z^{\mu_i - \lambda} v_i\right) \\ &= z^\lambda v \end{aligned}$$

and thus z acts by a scalar on $V(\lambda)$ and we obtain that the action of G on $\mathbb{P}(V(\lambda))$ factors through G_{ad}.

The faithfulness of the G_{ad} action is a consequence of 3): The G orbit of $[V_\lambda] \in \mathbb{P}(V(\lambda))$ is isomorphic to G/B, compare Remark 4. One can identify G/B with the set of all Borel subgroups of G, denoted with $\mathfrak{B}$. The action of G on G/B translates to $g \cdot B' = gBg^{-1}$ and the set of fixed points of a subgroup $H \subset G$ is $\mathfrak{B}^H = \{B' \in \mathfrak{B} | H \subset B'\}$. The faithfulness is now a consequence of the so called selfnormalizing theorem, i.e. for any

Borel group B' we have $B' = N_G(B')$. In the case of a semisimple group G we have

$$\cap_{B'\in\mathfrak{B}} B' = \cap_{T'\in\mathfrak{T}} T' = Z(G)$$

where $\mathfrak{T}$ is the set of maximal tori and thus G_{ad} acts faithfully on G/B. For the identification of G/B with $\mathfrak{B}$ see [5, Chapter 11.18] and for $\cap_{B'\in\mathfrak{B}} B' = \cap_{T'\in\mathfrak{T}} T' = Z(G)$ see [13].

□

Now let the situation be as above, we write $V(\lambda) = V$ for short. Consider $End(V) = V \otimes V^*$ and let $id \in End(V)$ denote the identity. We have a $G \times G$ action on $End(V)$ by $(g_1, g_2) \cdot A = g_1 A g_2^{-1}$. Let

$$\begin{aligned} diag : G &\rightarrow G \times G \\ g &\mapsto (g, g) \end{aligned}$$

be the diagonal embedding of G into $G \times G$. We have the following consequence of the above theorem.

Theorem 3.1.2. *The orbit* $(G \times G)[id]$ *in* $\mathbb{P}(End(V))$ *is isomorphic to the homogeneous space* $(G \times G)/((Z(G) \times Z(G))diag(G)) \simeq G_{ad}$.

Proof. An element in the $G \times G$ stabilizer of $[id]$ is a pair (g_1, g_2) such that $g_1 g_2^{-1}$ acts on V by a scalar. By Theorem 3.1.1 5) this in turn is equivalent to $g_1 g_2^{-1} \in Z(G)$ and this is equivalent to $(g_1, g_2) \in (Z(G) \times Z(G))diag(G)$ by the following observation: That $(Z(G) \times Z(G))diag(G)$ is contained in the stabilizer of $[id]$ is clear. For the reverse inclusion just observe that since $g_1 g_2^{-1} = z \in Z(G)$ we have $g_1 = z g_2$ which makes $(g_1, g_2) = (z g_2, g_2)$ an element of $(Z(G) \times Z(G))diag(G)$. □

3.2 The setup

In the following we fix a semisimple group of adjoint type and fix a maximal torus T. Let G_{sc} be the semisimple simply connected cover of G. Fix a maximal torus $T_{sc} \subset G_{sc}$ and let T be the image of T_{sc} in T which is again a maximal torus. We denote $Lie(G_{sc}) = Lie(G) = \mathfrak{g}$ and $Lie(T_{sc}) = Lie(T) = \mathfrak{t}$. Besides that the notation is as in Chapter 2.7. If we want to stress the context we will write a subscript *sc*, so for example the Borel subgroup corresponding to the positive roots will be denoted $B \subset G$ respectively $B_{sc} \subset G_{sc}$. The only exceptions from this rule will be the unipotent groups $U \subset G$, since by Theorem 2.7.2 we will have $U \simeq U_{sc}$ and we will therefore identify the two.

As in the first chapter we denote with R_+ the positive roots with respect to T and $\{\alpha_1, \ldots, \alpha_l\}$ the simple roots. By the remarks in chapter 2.9 we have the the characters of $X(T)$ correspond to the characters of $X(T_{sc})$ that are trivial on the center $Z(G_{sc})$. So if we have a character in $\chi \in X(T_{sc})$ that is trivial on the center we might and will compute $\chi(t) = t^\chi$ for $t \in T$.

Remark 15. Let λ be a regular weight of G_{sc} and denote throughout this chapter with $V(\lambda) = V$ the corresponding highest weight representation as in 3.1.1. Again by Theorem 3.1.1 we can choose a basis of weight vectors $v_0, \dots, v_n$ with the following properties:

1) v_0 has weight λ.
2) v_i has weight $\lambda - \alpha_i$ for $0 < i \leq l$.
3) Let λ_i be the weight of v_i then if $\lambda_i < \lambda_j$ we have $i > j$.

Note that we have that $\dim(V_{\lambda_i}) = 1$ for $i = 0, \dots, l$.

Lemma 3.2.1. *We have that $U^- v_i - v_i$ is in the span of the v_j for $j > i$.*

Proof. By theorem 2.7.2 we have that $\exp : \mathfrak{n}^- \to U^-$ is an isomorphism and thus by Theorem 2.6.7 we have that $U^- v_i = v_i + \sum_{\lambda_j < \lambda_i} V_{\lambda_j}$ and the claim follows from the choice of basis last remark. □

Definition 3.2.1. Let V be as above for a regular dominant weight. Let $\mathbb{P}(V)$ be the associated projective space. By Theorem 3.1.1 we have that the action of G_{sc} on $\mathbb{P}(V)$ factors through an action of G. Define

$$\mathbb{P}_0(V) = \left\{ \left[\sum_{i=0}^{n} b_i v_i \right] | b_0 \neq 0 \right\} \simeq k^n$$

and notice that by Lemma 3.2.1 applied to v_0 we have that $\mathbb{P}_0(V)$ is U^- stable.

Lemma 3.2.2. *We have that the orbit $U^- \cdot [v_0]$ is a closed subvariety of $\mathbb{P}_0(V)$ isomorphic to U.*

Proof. We first observe that the stabilizer of $[v_0]$ in U^- is trivial, indeed by Theorem 3.1.1 this is just B and by Theorem 2.4.1 3) we have that $U^- \cap B$ is trivial.

The rest now follows directly from Theorem 2.7.3. □

Remark 16. Consider the dual representation V^*. G_{sc} acts on V^* by $g \cdot f = f \circ g^{-1}$. Choose a dual basis $v^0, \dots, v^n$ and note that v^i has weight $-\lambda_i$, so that v^0 is a highest weight vector if we reverse the choice of positive roots. Define

$$\mathbb{P}_0(V^*) = \left\{ \left[\sum_{i=0}^{n} b_i v^i \right] | b_0 \neq 0 \right\} \simeq k^n$$

Reversing the choice of positive roots the exact same arguments as above can be made for the orbit $U \cdot [v^0] \subset \mathbb{P}(V^*)$. We state this in the following lemma.

Lemma 3.2.3. *We have that the orbit $U \cdot [v^0]$ is a closed subvariety of $\mathbb{P}_0(V^*)$ isomorphic to U.*

Remark 17. We know return to the case of $G \times G$ again and consider the representation $V \otimes V^*$. We have that $V \otimes V^* \simeq End(V)$ as $G_{sc} \times G_{sc}$ representations and the action of $G_{sc} \times G_{sc}$ translates to $(g_1, g_2) \cdot A = g_1 A g_2^{-1}$. A basis is given by the vectors $\{v_i \otimes v^j | i, j = 0, \ldots, n\}$. As above the action of $G_{sc} \times G_{sc}$ on $\mathbb{P}(End(V))$ factors through $G \times G$. Consider the affine subspace of $\mathbb{P}(End(V))$ given by

$$\mathbb{P}_0 = \left\{ \left[\sum a_{ij} v_i \otimes v^j \right] | a_{00} \neq 0 \right\} = \left\{ [A] \in \mathbb{P}(End(V)) | v^0(A v_0) \neq 0 \right\}.$$

By the above considerations $U^- T \times U$ preserves $\mathbb{P}_0$. Next define a morphism

$$\begin{aligned} \varphi : G &\rightarrow \mathbb{P}(End(V)) \\ g &\mapsto [g] \end{aligned}$$

and notice that φ is $G \times G$ equivariant, where the action of $G \times G$ on G is given by $(g_1, g_2) \cdot g = g_1 \cdot g \cdot g_2^{-1}$. Now let $X = \overline{\varphi(G)}$ be the closure of the image of G under this morphism. X is the compactification of G that we are interested in.

3.3 The geometry of the affine part

We will now investigate the geometry of $X_0 = X \cap \mathbb{P}_0$ and later use this information to make statements about the global geometry. In the following denote the Weyl group $W = N_G(T)/T$ and for $w \in W$ denote with $\overline{w} \in N_G(T)$ a representative of w in $N_G(T)$.

Lemma 3.3.1. *Let everything be as above. Then*

$$X_0 \cap \varphi(G) = \varphi(U^- T U).$$

Proof. We have that $[id] = [\sum v_i \otimes v^i]$ is in $\mathbb{P}_0$ and therefore $\varphi(e) = [id]$ is in X_0. Now by the $U^- T \times U$ invariance of $\mathbb{P}_0$ we have that $\varphi(U^- T U) = U^- T \times U \cdot \varphi(e) \subset X_0$.

For the other direction recall that by the Bruhat decomposition 2.8.2 we have $G = \cup_{w \in W} w U^- T U$. We will show that $w U^- T U$ takes the identity to the complement of $X_0 \cap \varphi(G)$. We have already seen that $U^- T U$ leaves $[v_0 \otimes v^0]$ invariant, by theorem 2.8.1 we know that $\overline{w} v_\lambda$ is a weight vector of weight $w(\lambda)$. But since λ is regular we have by Theorem 2.8.3 that if $w \neq id$ then $w(\lambda) \neq \lambda$. We conclude that for $w \neq id$ we have that $\overline{w} \cdot [v_0 \otimes v^0] \notin \mathbb{P}_0$ and thus $\varphi(\overline{w}) \notin X_0$. But now since X_0 is $U^- T \times U$ invariant so is its complement and the result follows. □

Remark 18. By Theorem 2.8.2 $U^- T U$ is dense in G and we have that $\varphi(U^- T U)$ is dense in X_0 and thus we have that X_0 is the closure $\overline{\varphi(U^- T U)}$ in $\mathbb{P}_0$.

Lemma 3.3.2. *Let $Z = \overline{\varphi(T)}$ be the closure of $\varphi(T)$ in $\mathbb{P}_0$. We have that $Z = k^l$ where $l = dim(T)$.*

Proof. Recall that the vectors v_i have weight $\lambda_i = \lambda - \alpha_i$ for $i = 1, \ldots, l$ and weight $\lambda_j = \lambda - \sum n_{kj}\alpha_k$ in general. The identity $e \in G$ gets send to $\varphi(e) = [id] = [\sum v_i \otimes v^i]$. Observe that under the isomorphism $End(V) \simeq V \otimes V^*$ if $A \simeq \sum a_{ij} v_i \otimes v^j$ we have for any $g \in G$ that $g \cdot A \simeq \sum a_{ij}(g \cdot v_i) \otimes v^j$. As remarked in the proof of Theorem 3.1.1 we have that $\lambda_i - \lambda$ is trivial on the center. Choosing a representative $\bar{t} \in T_{sc}$ we thus get for $t \in T$:

$$\begin{aligned}
\varphi(t) = t \cdot \varphi(e) &= \left[\sum (\bar{t} \cdot v_i) \otimes v^i\right] \\
&= \left[\sum \bar{t}_i^{\lambda} v_i \otimes v^i\right] \\
&= \left[v_0 \otimes v^0 + \sum t^{\lambda_i - \lambda} v_i \otimes v^i\right] \\
&= \left[v_0 \otimes v^0 + \sum_{i=1}^{l} t^{-\alpha_i} v_i \otimes v^i + \sum_{j=l+1}^{n} t^{-\sum n_{jk}\alpha_k} v_j \otimes v^j\right]
\end{aligned}$$

Now define $F : k^l \to \mathbb{P}_0$ by

$$F(z_1, \ldots, z_l) = \left[v_0 \otimes v^0 + \sum_{i=1}^{l} z_i v_i \otimes v^i + \sum_{j=l+1}^{n} \prod_{i=1}^{l} z_i^{n_{ji}} v_k \otimes v^k\right].$$

We have that F is a closed embedding and moreover that $\varphi(T)$ is a open subset of the image of F given by the complement to the equation $\prod z_i = 0$. Thus we have $Z = \overline{\varphi(T)} = F(k^l) \simeq k^l$ and the claim follows. □

Remark 19. The next theorem will proof an isomorphism of X_0 with $k^{dim(G)}$. To this end recall that $U^- \times U$ leaves X_0 invariant and define $\phi : U^- \times U \times Z \to X_0$ by $\phi(u, v, z) = (u, v) \cdot z = u \cdot z \cdot v^{-1}$. Here we have $Z = \overline{\varphi(T)}$ as above.

Theorem 3.3.3. *Let $\phi : U^- \times U \times Z \to X_0$ be defined as above. We have that ϕ is in isomorphism and thus X_0 is smooth and isomorphic to $k^{dim(G)}$.*

The proof will make use of the following lemma:

Lemma 3.3.4. *Let G be an algebraic group and Z a G variety. Regard G as a G variety via left multiplication. Now suppose there is a G equivariant morphism $f : Z \to G$. Let $F = f^{-1}(e)$, then $Z \simeq G \times F$.*

Proof of lemma. Define $\chi : A \times F \to Z$ by $\chi(a, z) = a \cdot z$. Furthermore define $\eta : Z \to F$ by $\eta(z) = f(z)^{-1} \cdot z$ and $\tau : Z \to A \times F$ by $\tau(z) = (f(z), \eta(z))$. Now we can check that $\chi \circ \tau$ and $\tau \circ \chi$ are the identity. The first equality being very straight forward and the second only using that $z \in F$ gets send to e via f. □

Proof of theorem 3.3.3 . Suppose for now there exists a morphism $\beta : X_0 \to U^- \times U$ with the properties

1) β is $U^- \times U$ equivariant.

2) $\beta(\phi(u, v, z)) = (u, v)$ for all $z \in \varphi(T)$.

Then using Lemma 3.3.4 by the $U^- \times U$ equivariance we get that

$$X_0 \simeq U^- \times U \times \beta^{-1}(e, e).$$

By property 2) of β we have that $\varphi(T) \subset \beta^{-1}(e, e)$ and since $\beta^{-1}(e, e)$ is closed we have $Z = \overline{\varphi(T)} \subset \beta^{-1}(e, e)$. Recall that $X_0 = \overline{\varphi(G)} \cap \mathbb{P}_0$ and so it has dimension at most $dim(G) = dim(U^-) + dim(U) + dim(T)$ and since $X = \overline{\varphi(G)}$ is irreducible so is X_0. Putting this together we obtain that $\beta^{-1}(e, e)$ is irreducible and has dimension maximal $dim(T) = dim(Z)$ and contains Z and thus $\beta^{-1}(e, e) = Z$.

We will now construct β. Define

$$\begin{aligned} F : \mathbb{P}_0 &\rightarrow \mathbb{P}_0(V) \\ [A] &\mapsto [Av_0] \end{aligned}$$

which is well defined since $[A] \in \mathbb{P}_0$ means by definition that $v^0(Av_0)$ is non-zero and therefore $[Av_0] \in \mathbb{P}_0(V)$. Call $\xi = F_{|X_0}$ the restriction of F to X_0. By Lemma 3.3.1 we have that $\varphi(U^-TU) \subset X_0$ and for $u \in U^-, t \in T, v \in U$ we have

$$\xi(\varphi(utv)) = utv[v_0] = u[v_0], \tag{3.1}$$

the last equality being true since $TU = B$ fixes $[v_0]$ (see the proof of Lemma 3.2.2). We thus have

$$\xi(\varphi(U^-TU)) = U^-[v_0]$$

which again by Lemma 3.2.3 is closed in $\mathbb{P}(V)$. We have that $\varphi(U^-TU)$ is dense in X_0 (see remark 18) and therefore we obtain

$$\xi(X_0) = U^-[v_0].$$

Consider the isomorphism $U^- \rightarrow U^-[v_0]$ as discussed in Lemma 3.2.2 and denote with $\eta : U^-[v_0] \rightarrow U^-$ its inverse. And define

$$\beta_1 = \eta \circ \xi : X_0 \rightarrow U^-$$

Furthermore since F and therefore ξ is U^- equivariant and the same is true for η we have that β_1 is U^- equivariant.

To define β_2 we consider the isomorphism

$$End(V) \simeq V \otimes V^* \simeq V^* \otimes V \simeq End(V^*)$$

where the second isomorphism is just switching the components. Call A^* the image of $A \in End(V)$ under this isomorphism we now obtain a morphism

$$\begin{array}{rcl} E : X_0 & \to & \mathbb{P}_0(V^*) \\ [A] & \mapsto & [A^* v^0]. \end{array}$$

Now reversing our choice of positive roots we can regard v^0 as a highest weight vector for V^* and we see that we can do all arguments similarly by replacing U^- with U and B with B^-. The restriction ζ of E to X_0 restricts to a morphism $\zeta : X_0 \to U[v^0]$ and taking η^* the inverse of the isomorphism $U \to U[v^0]$ we obtain

$$\beta_2 = \eta^* \circ \zeta : X_0 \to U.$$

Now $\beta : X_0 \to U^- \times U$ given by $\beta(x) = (\beta_1(x), \beta_2(x))$ is the desired map. Indeed β is equivariant and by equation 3.1 we have that $\beta_1(\phi(u, v, z)) = u$ for $z \in \varphi(T)$ and a similar statement holds for β_2. □

3.4 The global geometry

We will now use the smoothness of X_0 to show that X is smooth. We will use the following facts.

Lemma 3.4.1. *Let G be a semisimple algebraic group and V be an irreducible representation of G and v be a highest weight vector. Then the induced action of G on $\mathbb{P}(V)$ has a unique closed orbit $G \cdot [v]$.*

Proof. Let $v \in V$ by Theorem 2.7.4 we have that $G_{[v]}$ is parabolic if and only if v is a vector of highest weight. Therefore $G \cdot [v] \simeq G/G_{[v]}$ is closed if and only if v is a highest weight vector. □

Lemma 3.4.2. *If X is a G variety with a unique closed orbit Y and U is an open set in X with $U \cap Y \neq \emptyset$ then*

$$X = \bigcup_{g \in G} gU.$$

Proof. Let $x \in X$ and consider the orbit $G \cdot x$ of x. Since $U \cap Y \neq \emptyset$ and Y is the unique closed orbit we have $Y \subset \overline{G \cdot x}$ by the closed orbit lemma 2.3.1 and therefore $U \cap \overline{G \cdot x} \neq \emptyset$ and therefore since U is open $U \cap G \cdot x \neq \emptyset$. It follows that $x \in \bigcup_{g \in G} gU$. □

Lemma 3.4.3. *Let $W \subset X = \overline{\varphi(G)}$ be a $G \times G$-stable closed subvariety of X. Then*

$$W = \bigcup_{a \in G \times G} a \cdot (W \cap X_0).$$

Proof. Note that $End(V) \simeq V \otimes V^*$ is an irreducible representation of $G \times G$ and $v_0 \otimes v^0$ is a highest weight vector stabilized by $B \times B^-$. Thus by Lemma 3.4.1 $\mathbb{P}(End(V))$ has a unique closed $G \times G$ orbit through $p_\lambda = [v_0 \otimes v^0]$. Since W is $G \times G$ stable, it has a closed orbit, which must be $Y = (G \times G)p_\lambda$. Since $p_\lambda \in X_0 \cap Y$ we have that $X_0 \cap W$ is non-empty and since it is also open in W we can apply Lemma 3.4.2. □

Theorem 3.4.4. *The following holds:*

1) $X = \bigcup_{a \in G \times G} a \cdot X_0$ is smooth.

2) Let $Q \subset X$ be a $G \times G$ orbit. Then

$$\overline{Q} = \bigcup_{a \in G \times G} a \cdot (\overline{Q} \cap X_0).$$

3) If Q and Q' are two $G \times G$ orbits in X, and $\overline{Q} \cap X_0 = \overline{Q'} \cap X_0$ then we have $Q = Q'$.

Proof. For 2) apply Lemma 3.4.3 to $W = \overline{Q}$. In the case $Q = (G \times G) \cdot [id] = \varphi(G)$ we have $\overline{Q} = X$ and 1) follows from 2). To prove 3) note that by 2) we have $\overline{Q} = \overline{Q'}$ and since they are both open in their closure the result follows.

□

3.5 Description of the $G \times G$ orbits

Remark 20. We classify the $G \times G$ orbits in X and show they have smooth closure. Let $Z = \overline{\varphi(T)}$ as above which we identify with $k^l = (z_1, \ldots, z_l)$ by Lemma 3.3.2. Let $I \subset \{1, \ldots, l\}$ and denote

$$Z_I := \{(z_1, \ldots, z_l) \in Z | z_i = 0 \text{ for all } i \in I\}$$

and let

$$Z_I^0 := \{(z_1, \ldots, z_l) \in Z_I | z_j \neq 0 \text{ for all } j \notin I\}.$$

Then we have that Z_I is the closure of Z_I^0 and $Z_I \simeq k^{l-|I|}$ and $Z_I^0 \simeq (k^*)^{l-|I|}$ and by Lemma 3.3.2 we have that $T \simeq (k^*)^l$ acts on Z by $(a_1, \ldots, a_l) \cdot (z_1, \ldots, z_l) = (a_1 z_1, \ldots, a_l z_l)$ in appropriate coordinates and the Z_I^0 are precisely the $T \times \{e\}$ orbits in Z. Set $z_I = (z_1, \ldots, z_l)$ with $z_i = 1$ if $i \notin I$ and $z_i = 0$ if $i \in I$. Then

$$Z_I = (T \times \{e\}) z_I.$$

For $i = 1, \dots, l$ we have $Z_i := Z_{\{i\}}$ is a hypersurface and $Z_I = \cap_{i \in I} Z_i$. Let

$$\Sigma_I = \phi(U^- \times U \times Z_I)$$

and

$$\Sigma_I^0 = \phi(U^- \times U \times Z_I^0).$$

Proposition 3.5.0.1. *The Σ_I^0 are precisely the $U^-T \times U$ orbits in X_0, and the closure Σ_I in X_0 is isomorphic to $k^{dim(G)-|I|}$. In particular $U^-T \times U$ has precisely 2^l orbits in X_0 and all these orbits have smooth closure.*

Proof. Recall that ϕ is an isomorphism and by definition $\phi(u, v, z) = uzv^{-1}$ which is $U^- \times U$ equivariant. Now let $x \in X_0$, we want to examine the orbit $(U^-T \times U)x$. Let (u, v, z) be the primage of x by ϕ. By the $U^- \times U$ equivariance $\phi(z) = z$ is in the same orbit as x. But now $(U^-T \times U)z$ is just $(U^- \times U) \cdot (T \cdot z)$ with preimage under ϕ equal to $U^- \times U \times T \cdot z$. The T orbits in Z are known and so $T \cdot z$ has to be one of the Z_I^0. □

Remark 21. Let Q be a $G \times G$ orbit in X, Q is irreducible by Theorem 2.3.2. Then $\overline{Q} \cap X_0$ is closed and irreducible in X_0 and $U^-T \times U$ stable, so it can not be a proper union of the finitely many orbit closures by its irreducibility and so it must be the closure of one of the 2^l $U^-T \times U$ orbits in X_0. By Theorem 3.4.4 3) it follows that $G \times G$ has at most 2^l orbits. We will now establish that there are in fact exactly 2^l orbits.

Lemma 3.5.1. *Let P be an irreducible projective variety with open affine subset U. Then all irreducible components of $P \setminus U$ have codimension 1 i.e. are divisors.*

Proof. See [14, Chapter 2] □

Lemma 3.5.2. *We have that $X \setminus \varphi(G) = \cup_{i=1,\dots,l} S_i$, where*

1) *S_i is a $G \times G$ stable divisor.*
2) *$S_i = \overline{\Sigma_i}$.*
3) *$S_i \cap X_0 = \Sigma_i$.*

Proof. Let $X \setminus \varphi(G) = \cup_\alpha S_\alpha$ be the decomposition of $X \setminus \varphi(G)$ into irreducible components. By Theorem 3.1.2 $\varphi(G) \simeq G$ we have that $\varphi(G)$ is affine and thus by Lemma 3.5.1 we have that S_α is a divisor.

Since $X \setminus \varphi(G)$ is $G \times G$ stable and $G \times G$ is connected it follows that the image of S_α under the $G \times G$ action is a irreducible set that contains S_α and so it must be equal to S_α. Therefore each S_α is $G \times G$ stable. It follows that $S_\alpha \cap X_0$ is $U^-T \times U$ stable and is a closed irreducible hypersurface in X_0. But X_0 has finitely many $U^-T \times U$ orbits and therefore $S_\alpha \cap X_0$ can not be a proper union of orbit closures by its irreducibility. Hence it must be one of the Σ_i.

Conversely, the closure of Σ_i in X is a closed irreducible hypersurface. Since $X_0 \cap \varphi(G) = \Sigma_\emptyset$ by Lemma 3.3.1 we have that Σ_i is contained in the closed set $X \setminus \varphi(G)$ and therefore $S_i = \overline{\Sigma_i}$ is contained in $X \setminus \varphi(G)$. This means we can identify $S_\alpha = S_i$ for some α and $S_\alpha \cap X_0 = \Sigma_i$. The result follows with Theorem 3.4.4. □

Lemma 3.5.3. *The following holds:*

1) *Let $S_I = \cap_{i\in I} S_i$. Then $S_J \subset S_I$ if $I \subset J$ and $S_I \cap X_0 = \Sigma_I$.*
2) *Let $S_I^0 = S_I \setminus \cup_{I \subsetneq J} S_J$. Then $S_I^0 = (G \times G) z_I$ is a single $G \times G$ orbit and $S_I^0 = S_J^0$ implies $I = J$.*
3) *$S_I = \cap_{a \in G\times G} a\Sigma_I$ and in particular S_I is smooth.*

Proof. The first claim of 1) follows from the definition and the second claim from Lemma 3.5.2 3). For 2) it is clear that $S_I^0 \cap X_0 = \Sigma_I^0$ by 1) and the description of the Σ_I^0. Since the S_i are $G \times G$ stable, so is S_I and therefore S_I^0. Let $x, y \in S_I^0$. By Proposition 3.4.4 1) there are $a, b \in G \times G$ s.t. $a \cdot x$ and $b \cdot y$ are in X_0 and thus ax, by are in Σ_I^0. By Proposition 3.5.0.1 there is a $c \in U^-T \times U$ such that $ax = cby$. Now part 2) follows since $z_I \in S_I^0$. Part 3) follows from Proposition 3.4.4 2) and the fact that the Σ_I are smooth. □

Remark 22. Let X be smooth variety with hypersurface Z. We say Z is a divisor with transversal crossing at $x \in Z$ if there an open neighbourhood U of x such that $Z \cap U = D_1 \cap \ldots \cap D_k$ is a union of hypersurfaces and $D_{i_1} \cap \ldots \cap D_{i_j}$ is smooth of codimension j for each tuple $\{i_1, \ldots, i_j\} \subset \{1, \ldots, k\}$. The standard example is to take $X = k^n$ and Z to be the variety given by the vanishing of $z_1 \cdot \ldots \cdot z_k$. Then Z is the union of the hyperplanes given by vanishing z_i.

Theorem 3.5.4. *$G \times G$ has 2^l orbits given by $S_I^0 = (G \times G) z_I$. In particular all orbits have smooth closure and $X \setminus \varphi(G)$ is a divisor with transversal crossing.*

Proof. Let Q be a $G \times G$ orbit in X. Then $\overline{Q} \cap X_0 = \Sigma_I$ by Remark 21 and Proposition 3.5.0.1. Since Σ_I is the closure of Σ_I^0 in X_0 it follows that $\Sigma_I^0 \subset Q$ and so $S_I^0 = Q$ by Lemma 3.5.3 2). By the same Lemma there are exactly 2^l orbits. Since Σ_I is the closure of Σ_I^0 it follows that S_I is the closure of S_I^0 using Lemma 3.4.2. From Remark 20 it is clear that $X_0 \setminus \varphi(U^-TU)$ is a divisor with transversal crossing. Now the last part follows from Proposition 3.4.4. □

We summarize the above analysis in the following theorem.

Theorem 3.5.5 (Compactification of G)**.** *Let G be a semisimple adjoint group and let $G \times G$ act on G by conjugation. Let l be the dimension of a maximal torus in G. Then there is a projective variety X with the following properties:*

1) *X has an open dense $G \times G$ orbit isomorphic to G.*
2) *X is smooth with finitely many $G \times G$ orbits.*
3) *The orbit closures are all smooth.*

4) There is a 1-1 correspondence between the set of orbit closures and subsets of $I = \{1, \ldots, l\}$. *If* $J \subset I$ *we denote* S_J *the corresponding orbit closure.*

5) It holds that $S_J \cap S_K = S_{J \cup K}$ *and* $codim(S_J) = |J|$.

6) Each S_J *is the transversal crossings of the* S_i *with* $i \in J$.

Remark 23. As a final remark it is worth noting that the above construction is indeed independent of the choice of regular dominant highest weight. A proof of this fact can be found in C. de Concini and C. Procesi [1] or Sam Evens and Benjamin F. Jones [2].

The construction of the wonderful compactification can be carried out for an algebraically closed field of positive characteristic too. In fact the construction is completely similar and only the choice of a suitable representation becomes more difficult. It can be done via the so called Steinberg modules and is carried out in [18].

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